Matrices of Infinite Series: Checking Non-Singularity Based on Truncations

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In [1] an algorithm is proposed that is applicable to an arbitrary non-singular square matrix $P = (p_{ij})$, in which all entries p_{ij} are polynomials in x over a field K; the algorithm allows one to check whether the matrix $P + Q$ is non-singular for any matrix Q with entries from the ring $K[[x]]$ of formal power series that, first, is of the same size as P and, second, has the property

$$\mathrm{val}\ Q \geqslant 1 + \deg P. \tag{1}$$

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[1] Abramov S., Barkatou M. On Strongly Non-Singular Polynomial Matrices// In: Advances in Computer Algebra. Springer Proceedings in Mathematics & Statistics, vol 226, 2018, pp. 1–17,

Recall that the valuation $\mathrm{val}\, f(x)$ of a power series or polynomial is the lowest power of x that has a nonzero coefficient in $f(x)$ (for example, for $f(x) = -3x^2 + 5x^3 + \dots$ we have $\mathrm{val}\, f(x) = 2$; by definition $\mathrm{val}\, 0 = \infty$). The valuation of a matrix composed of power series or polynomials is the smallest of the valuations of all entries of this matrix. The degree of a matrix composed of polynomials is considered to be the largest of all the degrees of the entries, wherein $\deg 0 = -\infty$.

A matrix P for which the matrix $P + Q$ is non-singular whenever for a matrix Q of the same size as P, the inequality (1) is satisfied, is called *strongly non-singular*. A strongly non-singular matrix remains non-singular when we add "tails" to its entries, turning the polynomials into series with coefficients in K. When adding "tails", it is necessary that the lowest power (valuation) of each such "tail" exceed $\deg P$.

Let a matrix P be non-singular and be obtained by truncating some matrix

$$M = (m_{ij}), \tag{2}$$

whose entries are formal power series, and the degree of truncation for all entries is equal to a fixed non-negative integer d: all terms of higher degrees than d are removed.

Then, if $d = \deg P$, using the algorithm proposed in [1],we can check whether the matrix M (we do not know this matrix) cannot be singular. For this purpose, the strong non-singularity of the matrix P is checked.

Remark 1

If d is greater than a power of some p_{ij}, then it is implied that the coefficients at the powers $\deg p_{ij} + 1, \ldots, d$ in m_{ij} are equal to zero.

We further consider the problem of checking the strong non-singularity in a more general version, in comparison with [1], by expanding the very concept of the strong non-singularity, omitting the assumption that the degrees of truncations of all entries of the original matrix are the same and, in particular, assuming that the degree of any (and even each) entry of the matrix P can be less than d — see Remark 1.

Below, P is a non-singular $n \times n$-matrix whose entries are polynomials in x over a field K. (The degrees of these polynomials may differ from each other.)

Preliminaries

In [1] the criterion was proved, i.e. a necessary and sufficient condition for the strong non-singularity of the matrix P:

Proposition 1

Let $P = (p_{ij})$, $p_{ij} \in K[x]$, be a non-singular polynomial $n \times n$-matrix and let d be an integer such that $d \geqslant \deg p_{ij}$, $i, j = 1, \ldots, n$. Then the matrix $P + Q$ is non-singular for any $n \times n$-matrix Q with entries from the ring $K[[x]]$, possessing property $\mathrm{val}\ Q \geqslant d + 1$ if and only if

$$d + \mathrm{val}\, P^* \geqslant \mathrm{val}\ \det P, \tag{3}$$

where P^* is the cofactor matrix of P.

This criterion will be used substantially further in the algorithm for solving the problem under consideration.

For a field K, we assume that there exists an algorithm that checks the consistency of an arbitrary given polynomial system of equations over K, provided that the coefficients of the system belong to K and the system has a solution with components in K. In the case of an algebraically closed field, such an algorithm is based on Gröbner bases.

Together with the matrix P, the integers $d_{ij} \geqslant -1$, $i, j = 1, \ldots, n$ are considered given. Let

$$P = (p_{ij}), \quad p_{ij} \in K[x], \quad \deg p_{ij} \leqslant d_{ij}, \quad d = \max_{i,j} d_{ij}. \tag{4}$$

With respect to the entry m_{ij} of the matrix (2), it is assumed that $m_{ij} = p_{ij} + q_{ij}$, where $q_{ij} \in K[[x]]$ is some series such that $\mathrm{val}\, q_{ij} \geqslant d_{ij} + 1$. The equality $d_{ij} = -1$ means that we have no information about the entry m_{ij} of the matrix M, except for its membership in $K[[x]]$.

Example 1

Let the matrix $\begin{pmatrix} 2x + x^2 & 0 \\ 0 & x \end{pmatrix}$ and the truncation degrees $d_{11} = 2$, $d_{12} = -1$, $d_{21} = 2$, $d_{22} = 1$ be given. These data define the truncated matrix $\begin{pmatrix} 2x + x^2 + O(x^3) & O(x^0) \\ O(x^3) & x + O(x^2) \end{pmatrix}$. The notation $O(x^k)$ is used for some (unspecified) formal series, whose valuation is greater than or equal to k.

This problem of checking the non-singularity of a matrix M is more difficult than that considered in [1], where none of the added terms can have a degree less than $d+1$. Thus, the added term does not affect the initial terms of the determinant.

The solution of this new problem is reduced below to a series of consistency checks (in other words, — checks for the presence of solutions, solvability) of systems of polynomial equations.

We assume that among the integers d_{ij} there are non-negative ones. Thus, $d \geqslant 0$ in (4).

If all d_{ij} are equal to each other, then the matrix P will be called *flat*.

Clarification of the concept of the strong non-singularity of a matrix

Definition 1

A matrix P satisfying conditions (4) will be called *strongly non-singular* if for any $n \times n$-matrix $Q = (q_{ij})$ whose entries belong to $K[[x]]$, $\operatorname{val} q_{ij} \geqslant d_{ij} + 1$, the matrix $P + Q$ is non-singular. (If $d_{ij} = -1$, then q_{ij} can be any element of $K[[x]]$, and for any q_{ij} the matrix $P + Q$ is non-singular.)

It will be shown that for a given matrix P, it is possible to test algorithmically whether this matrix is strongly non-singular.

Definition 2

Let (4) hold. For a matrix P, its K-*completion* is the matrix obtained by adding to each p_{ij} for $d_{ij} < d$ some monomials of degrees $d_{ij} + 1, \dots, d$ with coefficients from K. *Formal completion* is a completion in which all added monomials have different indefinite (symbolic) coefficients.

It is not difficult to prove the following Proposition

Proposition 2

A matrix P is strongly non-singular if and only if any of its K-completions (each of which is a flat matrix) is strongly non-singular.

Example 2

For the matrix from Example 1 (slide 8), we get $d = 2$ and

$$\tilde{P} = \begin{pmatrix} 2x + x^2 & t_1 + t_2 x + t_3 x^2 \\ 0 & x + t_4 x^2 \end{pmatrix}$$

is its formal completion where t_1, t_2, t_3, t_4 are unknowns. We get

$$\det \tilde{P} = 2x^2 + (2t_4 + 1)x^3 + t_4 x^4.$$

The valuation of $\det \tilde{P}$ doesn't depend on the unknowns. Then this valuation is equal to 2 for any K-completions and, since $d = 2$, condition (3) holds for any K-completions. Then the given matrix is strongly non-singular.

Example 3

Now consider a more complex case.
For the truncated matrix

$$P = \begin{pmatrix} x + O(x^3) & O(x^2) \\ 1 + O(x^3) & x + x^2 + O(x^3) \end{pmatrix},$$

we get $d = 2$.
Then

$$\tilde{P}(t) = \begin{pmatrix} x & tx^2 \\ 1 & x + x^2 \end{pmatrix}$$

(t is an unknown), and $\det \tilde{P}(t) = x^3 + (-t + 1)x^2$. In this case, the valuation of $\det \tilde{P}(t)$ depends on t. Then this valuation is equal to 2 or 3 for different K-completions and condition (3) holds or does not hold. To check the strong non-singularity of P, we have to consider different K-completions.

In what follows, considering statements that are true for all solutions of some polynomial system of equations, we assume that if the system is empty (does not contain equations), then any set of values of the unknowns is its solution.

Theorem 1

Let $p(x;\, t_1, \dots, t_m)$ be a polynomial in x whose coefficients are polynomials over K in unknowns $t_1, \dots, t_m$. Let S be a polynomial over K consistent system of equations (possibly empty) with respect to unknowns $t_1, \dots, t_m$. There exists an algorithm (let us call it $\mathcal{A}$) that allows one to find all values of $\operatorname{val} p$ that are realized (arise) for some values $t_1, \dots, t_m \in K$ that satisfy the system S. Together with each found value ν of the valuation, this algorithm finds a consistent system S_ν of polynomial over K equations, the solutions of which are all those solutions of the system S for which $\operatorname{val} p = \nu$. The result of the algorithm is a list of conditional valuations *of the form $(\nu,\, S_\nu)$.*

Checking the strong non-singularity of a matrix

Algorithm $\mathcal{A}$ and criterion (3) lead to an algorithm for solving the main problem. The idea is as follows: using $\mathcal{A}$, for the formal completion $\tilde{P}$ of the matrix P find all pairs

$$\mathrm{val}\ \det \tilde{P},\ \mathrm{val}\,(\tilde{P})^*, \qquad (5)$$

that are realized for certain specific K-completions of the original matrix; for each of these pairs, check the satisfying of inequality (3). This may indicate the existence of a K-completion of P that is a singular matrix, or a matrix for which (3) does not hold. Then P is not a strongly non-singular matrix. If this does not happen, then any K-completion yields a strongly non-singular matrix, and by Proposition 2, P is strongly non-singular.

Example 4

For P from Example 3, we get val $\det \tilde{P}(t) = 2$ if $t \neq 1$ (since $d = 2$, conditions (3) holds). Otherwise, we get K-completion
$\tilde{P}(1) = \begin{pmatrix} x & x^2 \\ 1 & x + x^2 \end{pmatrix}$ with $\det \tilde{P}(1) = x^3$, val $\det \tilde{P}(1) = 3$. The cofactor matrix for $\tilde{P}(1)$ is $\begin{pmatrix} x + x^2 & -1 \\ -x^2 & x \end{pmatrix}$, its valuation is equal to 0.
Conditions (3) doesn't hold for $\tilde{P}(1)$, and therefore P is not strongly non-singular.
Above, in Example 2, the case of a strongly non-singular matrix was considered.

The algorithm for checking the strong non-singularity of a matrix with elements in the form of truncated series has been implemented by us in the environment of the computer algebra Maple 2025 as the *StronglyNonSingular* procedure. The Maple library containing this procedure, as well as the Maple session with examples of using the procedure, are available at

`http://www.ccas.ru/ca/_media/StronglyNonSingularSeriesMatrix.zip`.

To determine the consistency of polynomial over $\mathbb{Q}$ systems that arise during the operation of the procedure, the *Solve* procedure of the *Groebner* package built into Maple is used.