Searching for rational solutions of linear ordinary difference and differential systems: schemes equipped with checkpoints

S. Abramov, D.Khmelnov, A.Ryabenko

It is quite common that search algorithms for those solutions of difference and differential equations and systems that belong to a fixed class of functions are designed so that nonexistence of solutions of the desired type is detected only in the last steps of the algorithm.

In some cases, performing additional tests on the intermediate results makes it possible to stop the algorithm as soon as these tests imply that no solutions of the desired type exist.

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Let K be a field of characteristic 0.

The ring of polynomials and the field of rational functions of x are conventionally denoted as $K[x]$ and $K(x)$, respectively.

If R is a ring (in particular, a field), then $\mathrm{Mat}_m(R)$ denotes the ring of $m \times m$-matrices with entries from R.

In this talk, we consider systems of the form

$$A_r(x)\sigma^r y(x) + \cdots + A_1(x)\sigma y(x) + A_0(x)y(x) = 0 \tag{1}$$

where $\sigma y(x) = y(x+1)$, and $A_i(x)$, $i = 0, 1, \ldots, r$, are matrices belonging to $\mathrm{Mat}_m(K[x])$;

$A_r(x)$ is the *leading* matrix (we suppose that it is non-zero), and $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T$ is a column of unknown functions (T denotes transposition). If $k = \min\{l \mid A_l \neq 0\}$ then A_k is the *trailing* matrix of (1).

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The system (1) can be written in the form

$$L(y) = 0 \tag{2}$$

where

$$L = A_r(x)\sigma^r + \cdots + A_1(x)\sigma + A_0(x). \tag{3}$$

A solution $y(x) = (y_1(x), y_2(x), \ldots, y_m(x))^T \in K(x)^m$ of (2) is called a *rational* solution. If $y(x) \in K[x]^m$, it is called a *polynomial* solution (a particular case of a rational solution).

The problem of finding rational solutions of full-rank systems

$$A_r(x)\sigma^r y(x) + \cdots + A_1(x)\sigma y(x) + A_0(x)y(x) = 0$$

in the case where the matrix $A_r(x)$ may be singular, was considered in

📄 Abramov S.A., Khmelnov D.E. Denominators of rational solutions of linear difference systems of arbitrary order. *Program. Comput. Softw.* 38, 2 (2012), 84 – 91.

This algorithm is based on finding a *universal denominator* of rational solutions to the original system (for brevity, we call it the universal denominator for the original system), i.e., a polynomial $U(x) \in K[x]$ such that, if the system has a rational solution $y(x) \in K(x)^m$, then it can be represented as $\frac{1}{U(x)} z(x)$, where $z(x) \in K[x]^m$.

If a universal denominator $U(x)$ is known, we can make the substitution

$$y(x) = \frac{1}{U(x)} z(x) \qquad (4)$$

where $z(x) = (z_1, (x) \ldots, z_m(x))^T$ is a vector of new unknowns, and then apply one of the algorithms for finding polynomial solutions.

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Other approaches are also possible. For example, the approach presented in

📄 Abramov S.A. Search of rational solutions to differential and difference
systems by means of formal series. *Program. Comput. Softw. 41, 2 (2015),
65 – 73.*

is based on expanding a general solution of the original system (2) into a
series. After multiplication by a universal denominator $U(x)$, such a series
becomes to be a polynomial.

The notions of *induced operators* and *induced systems* are quite important for finding solutions of different kinds. The idea is that if the solution $y(x)$ is represented as a double-sided series in some powers of x (maybe like factorial powers), then the two-sided sequence $v = (v_n)$ of coefficients of this series satisfies the induced system $L^{\circledcirc}(v) = 0$, if and only if, the corresponding series satisfies the original system $L(y) = 0$. Here $L^{\circledcirc}$ is the induced operator for the operator L.

Each formal series by factorial powers will have the form

$$s(x) = \sum_{n \in \mathbb{Z}} v_n x^{\overline{n}}, \tag{5}$$

where

$$x^{\overline{n}} = \begin{cases} x(x-1)\ldots(x-n+1), & \text{if } n > 0, \\ 1, & \text{if } n = 0, \\ \frac{1}{(x+1)(x+2)\ldots(x+|n|)}, & \text{if } n < 0. \end{cases} \tag{6}$$

It has been shown by D.Khmelnov in

📄 Khmelnov D.E. Basis selection in solving linear functional equations. *Program. Comput. Softw. 28, 2 (2002), 102–105.*

that the map

$$x \rightarrow n + \sigma^{-1}, \ \ \sigma \rightarrow (n+1)\sigma + 1 \tag{7}$$

transforms an operator $L \in K^m [x, \sigma]$ into $L^{\textcircled{1}} \in K^m [n, \sigma, \sigma^{-1}]$.

The algorithms of EG family

📄 Abramov S.A. EG-eliminations. *J. Diff. Equ. Appl. 5 (1999), 4 – 5.*
allow to transform, e.g., $L^{\textcircled{i}}$ to the form, where the leading or the trailing
matrix is non-singular. This gives, resp., the operators $^+L^{\textcircled{i}}$ and $^-L^{\textcircled{i}}$.
Let $^-L^{\textcircled{i}}$ be of the form

$$B_l(n)\sigma^l + \cdots + B_t(n)\sigma^t, \quad l > \cdots > t$$

($B_t(n)$ is non-singular, B_l is non-zero). Then l and t are the *leading* and
trailing orders of the operator $^-L^{\textcircled{i}}$.

The determinant (a polynomial in n) of the matrix $B_t(n-t)$ we will call the *indicial polynomial* for L at infinity and denote this polynomial by $I_{L,\infty}(n)$. Correspondingly, $I_{L,\infty}(n) = 0$ is the *indicial equation* for L at infinity. If a series

$$s(x) = \sum_{n \leq k} v_n x^{\overline{n}}$$

represents a solution to the system $L(y) = 0$ and $v_k \neq 0$, k (the *valuation* of the series $s(x)$) is a root of the algebraic equation $I_{L,\infty}(n) = 0$.

A scheme provided with checkpoints is as follows (the symbol • marks a checkpont and a test after which the algorithm can be stopped):

1. Find $^-L^{①}$ and $I_{L,\infty}(n)$ • {If $I_{L,\infty}(n)$ has no integer root then STOP.}
   Let n^* be the largest integer root of the polynomial $I_{L,\infty}(n)$.

2. [Let $n^* < 0$. Let some intermediate stage of the algorithm allow to get quickly a number u such that $u \geq \deg U(x)$. • { If $n^* + u < 0$, then STOP without finishing the computation $U(x)$.}]
   Find a universal denominator $U(x)$. • {If $n^* + \deg U(x) < 0$, then STOP}.

3. Using $^-L^{\text{①}}$ and $U(x)$ compute the polynomial $P(x) \in K[x]^m$ which is the numerator of the rational solution of the initial system. So, $\frac{1}{U(x)}P(x)$ is the solution of $L(y) = 0$. Here, finding a solution

$$R(x) = \sum_{n \leq n^*} v_n x^{\overline{n}}$$

for the operator L, is combined with multiplying $R(x)$ by $U(x)$.

• {If in computing values of the "arbitrary constants" all the coefficients of $U(x)R(x)$ vanish for non-negative powers then STOP}.

The presented algorithm was implemented in Maple 2020 as a procedure `RationalSolution`.

The first argument of the procedure is a full rank system. The system is specified as a linear equation with matrix coefficients. Elements of a matrix coefficient are rational functions of one variable (for example, x) over the rational number field. For example,

$$
> S := \begin{bmatrix} x^2 + 102\,x + 101 & x^3 + 104\,x^2 + 305\,x + 202 \\ x^2 - x - 2 & x^3 + x^2 - 4\,x - 4 \end{bmatrix} \cdot y(x+2) + \begin{bmatrix} -x^2 - 99\,x + 202 & -x + 2 \\ x - 2 & \dfrac{x-2}{x+101} \end{bmatrix} \cdot y(x+1)
$$

$$
+ \begin{bmatrix} -x - 101 & -\dfrac{x+101}{x+100} \\ -x & -\dfrac{x}{x+100} \end{bmatrix} \cdot y(x) = 0 :
$$

$>$

The second argument of the procedure is a name of a vector of unknowns (for example, $y(x)$).

The third argument is optional. It is

- 'earlyterminate' = true, or
- 'earlyterminate' = false.

The default is 'earlyterminate' = true.

For 'earlyterminate' = true, the presented algorithm with checkpoints is used.

For 'earlyterminate' = false, the algorithm from

📄 Abramov S.A., Khmelnov D.E. Denominators of rational solutions of linear difference systems of arbitrary order. *Program. Comput. Softw.* 38, 2 (2012), 84 – 91.

is used.

Applying the procedure `RationalSolution` to S we get that S has no rational solutions, the empty list is returned.

```
> st := time( ) :
  RationalSolution(S, y(x), earlyterminate = false);
  time( ) − st;
```
$$[\ ]$$
$$3.060 \tag{1}$$
```
>
```

The computing time with the argument `'earlyterminate'` = false is 3.060 sec.

```
> st := time( ) :
  RationalSolution(S, y(x), earlyterminate = true);
  time( ) − st;
```
$$[\ ]$$
$$0.016 \tag{2}$$
```
>
```

The computing time with the argument `'earlyterminate'` = true is 0.016.

Here is a system which has non-trivial rational solutions.

$$
\begin{aligned}
> S_2 := & \begin{bmatrix} 0 & 0 \\ x^3 + 5x^2 + 9x + 5 & x^3 + 5x^2 + 9x + 5 \end{bmatrix} \cdot y(x+2) + \begin{bmatrix} 2x^2 - 2 & \dfrac{2\left(x^2 - 1\right)}{x + 101} \\ x^3 - x^2 - x + 1 & \dfrac{x^3 - x^2 - x + 1}{x + 101} \end{bmatrix} \cdot y(x+1) \\
& + \begin{bmatrix} -2x^2 + 2x & -\dfrac{2x\left(x - 1\right)}{x + 100} \\ -2x^3 + x^2 - 2x - 1 & -\dfrac{x^4 + 102x^3 + 99x^2 + 102x + 100}{x + 100} \end{bmatrix} \cdot y(x) :
\end{aligned}
$$

>

The procedure `RationalSolution` builds a basis of rational solutions linear space, returns a list of basis elements:

```
> st := time( ) :
  RationalSolution(S_2, y(x), earlyterminate = false);
  time( ) − st;
```

$$\left[\left[\begin{array}{c} -\dfrac{1}{(x+99)\left(x^2+1\right)} \\[2ex] \dfrac{x+100}{(x+99)\left(x^2+1\right)} \end{array}\right], \left[\begin{array}{c} -\dfrac{x^3+100\,x^2-59600\,x+100}{x\,(x+99)\left(x^2+1\right)} \\[2ex] \dfrac{x^3-59501\,x^2-5960099\,x+100}{x\,(x+99)\left(x^2+1\right)} \end{array}\right]\right]$$

$$60.840 \qquad\qquad \text{(3)}$$

```
> st := time( ) :
  RationalSolution(S_2, y(x), earlyterminate = true);
  time( ) − st;
```

$$\left[\left[\begin{array}{c} -\dfrac{1}{\left(x^2+1\right)(x+99)} \\[2ex] \dfrac{x+100}{\left(x^2+1\right)(x+99)} \end{array}\right], \left[\begin{array}{c} -\dfrac{x^3+100\,x^2-59600\,x+100}{x\left(x^2+1\right)(x+99)} \\[2ex] \dfrac{x^3-59501\,x^2-5960099\,x+100}{x\left(x^2+1\right)(x+99)} \end{array}\right]\right]$$

$$62.303 \qquad\qquad \text{(4)}$$

Our experiments show that in the absence of rational solutions, time savings are about 75%, and the additional time in the presence of solutions does not exceed 20-25%.

The implementation and a session of Maple with examples of using the procedure `RationalSolution` are available at the address http://www.ccas.ru/ca/lfs.