EG-eliminations as a tool for computing rational solutions of linear q-difference systems of arbitrary order with polynomial coefficients

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We consider linear q-difference systems with coefficients belonging to $\mathbb{K}[x]$, where $\mathbb{K} = K(q)$, K is a field of characteristic 0, and q is transcendental over K. A system is of the form

$$A_r(x)y(q^r x) + \cdots + A_1(x)y(qx) + A_0(x)y(x) = b(x), \qquad (1)$$

where

- $A_0(x), A_1(x), \ldots, A_r(x)$ are $m \times m$-matrices, whose elements belong to $\mathbb{K}[x]$ (we write: $A_0(x)$, $A_1(x)$, $\ldots$, $A_r(x) \in \mathrm{Mat}_m(\mathbb{K}[x])$), it is supposed that the matrices $A_0(x)$, $A_r(x)$ are non-zero,
- $b(x) = (b_1(x), \ldots, b_m(x))^T \in \mathbb{K}[x]^m$ is the right-hand side of the system,
- $y(x) = (y_1(x), \ldots, y_m(x))^T$ is an unknown column.

r is the *order* of system (1).

System (1) can be rewritten using the q-shift operator σ_q: $\sigma_q y(x) = y(qx)$. The matrices $A_r(x)$ and $A_0(x)$ are called the *leading* and, resp., the *trailing* matrices of system (1).

One of known computer algebra approaches to the search for solutions of linear systems is the cyclic vector method, which transforms a system into a scalar equation (a scalar equation can be considered as a system having only one equation involving one unknown), which is equivalent in a certain sense to the original system. Here the main problem is the overgrowth of the coefficients. This is the reason why this method works in general, for systems of small orders. This stimulates elaborating of direct algorithms which do not require preliminary cyclic vector method applying, or another type of uncoupling of a system.

In this talk we consider direct algorithms for constructing the solutions of a system having the form (1) with $y_1(x), \dots, y_m(x)$ belonging to the field $\mathbb{K}(x)$ of rational functions of x over $\mathbb{K}$. We call such solutions *rational*. If $y(x) \in \mathbb{K}[x]^m$, then this solution is *polynomial* (a particular case of rational solutions). Rational solutions may be a building block for other types of solutions, and more general, such algorithms may be a part of various computer algebra algorithms.

We will also consider solutions $y(x) = (y_1(x), y_2(x), \dots, y_m(x))^T \in \mathbb{K}((x))^m$ whose components are formal Laurent series (*Laurent* solutions).

We will suppose that equations of the original system are independent over $\mathbb{K}[x, \sigma_q, \sigma_q^{-1}]$ (i.e., the system is of *full rank*).
Various questions related the search for rational solutions both for the scalar equations and for systems were discussed earlier. For the q-difference case, some algorithms were proposed in, e.g., Abramov, 1995, 2002 for constructing all rational solutions of scalar linear equations and for first-order linear normal systems, i.e., the systems of the form

$$y(qx) = A(x)y(x), \tag{2}$$

where $A(x)$ is a non-singular (invertible in $\mathrm{Mat}_m(\mathbb{K}(x))$) matrix.
The possible singularity of the leading or the trailing matrices of (1) gives rise to interruptions of the search for solutions of a system (by the way, if the system (2) is rewritten as $I_m y(qx) - A(x)y(x) = 0$, where I_m is the identity $m \times m$-matrix then $-A(x)$ is the trailing matrix of the system.

The same can be said about the possible singularity of the leading or the trailing matrices of the so called *induced* recurrent (difference) system: a formal series $\sum a_n x^n$, $a_n \in \mathbb{K}^m$, satisfies the original q-difference system if and only if the sequence (a_n) of m-dimensional vectors satisfies the induced recurrent system. Below, we discuss the algorithm of EG-eliminations which allows to transform the original q-difference system and the induced recurrent system into systems having a non-singular leading or trailing matrix. After computing the determinant of the non-singular leading matrix one can find a lower bound for valuations of formal Laurent series solutions. An upper bound for degrees of polynomial solutions can be found using the non-zero determinant of the trailing matrix.

As for rational solutions, the search for them consists of two steps:
1) constructing a so called universal denominator or, in another terminology, a denominator bound, and
2) constructing the corresponding numerators of the components of the solutions.

The numerators mentioned in step 2 are the components of polynomial solutions of the system obtained from the original system by means of a special substitution on the base of the universal denominator constructed on step 1. Using the leading and the trailing matrices of the original system (possibly that after applying EG-eliminations) allows to construct the part of the universal denominator that contains only the factors other than x. Concerning the factors of the form x^k, it is remarked in Abramov, 2002 that a bound for k can be obtained when one considers rational solutions as Laurent series solutions at $x = 0$.

The first algorithm and an example of constructing polynomial solutions of q-difference systems of arbitrary order were given in Abramov, 1999. Concerning the universal denominators, note that strictly speaking the paper Abramov, 2002 is dedicated to first-order systems. However, in that paper, some general principals are formulated which allow to solve the problem in the case of higher order systems, by modifying algorithms for the difference case (such algorithms were published earlier). In Abramov, 2002, Abramov & Bronstein 2002 it is noted that for constructing the part of the universal denominator that contains only the factors other than x, it is reasonable to use the slightly modified version ($x + i \to xq^i$) of an algorithm for the difference case (such an algorithm for higher order difference systems was proposed in Abramov, Khmelnov, 2012). The treatment of rational solutions as Laurent ones to work with the factors x^k was considered also in Abramov, 2002.

In the present talk we follow this natural plan and obtain an algorithm for constructing rational solutions of systems of the form (1) (we suppose that the system is of full rank). More, we will mention another approach, transferring to the q-difference case the approach which was discussed for the difference case (Abramov, Gheffar, Khmelnov, 2010). For constructing polynomial solutions the algorithms from Abramov, 1999, Abramov& Bronstein, 2001, Khmelnov, 2004 can be used. Those algorithms are also based on EG-eliminations.

J.Middeke shown in 2017 that the Popov normal form can also be used for finding bounds for the exponent k and the degrees of polynomial solutions.

Embracing systems
Consideration of the so-called embracing systems allows us to avoid the assumption of invertibility of the matrices $A_r(x)$, $A_0(x)$.

For any system S of the form (1) one can construct an l-embracing system $\bar{S}$

$$\bar{A}_r(x)y(q^r x) + \cdots + \bar{A}_1(x)y(qx) + \bar{A}_0(x)y(x) = \bar{b}(x), \qquad (3)$$

with the leading matrix $\bar{A}_r(x)$ being non-singular (invertible), and with the set of solutions containing all the solutions of the system S. Similarly, one can construct a t-embracing system $\bar{\bar{S}}$

$$\bar{\bar{A}}_r(x)y(q^r x) + \cdots + \bar{\bar{A}}_1(x)y(qx) + \bar{\bar{A}}_0(x)y(x) = \bar{\bar{b}}(x), \qquad (4)$$

whose trailing matrix is non-singular (invertible), and with the set of solutions containing all the solutions of the system S. All the entries of the matrices and of the right-hand sides of (1) are in $\mathbb{K}[x]$. It is possible that the matrices $\bar{A}_0(x), \bar{\bar{A}}_r(x)$ are zero, either both or one of them.

The construction of the embracing systems can be performed with the algorithms EG (Abramov, 1999) or its improved version (Abramov& Bronstein, 2001).

The algorithm EG is applicable also to difference (recurrent) systems. Sequential solutions (i.e., solutions having the form of sequences) of such systems are interesting for us.

Induced recurrent systems

A formal Laurent series $\sum a_n x^n$, $a_n \in \mathbb{K}^m$ satisfies original q-difference system (1), if and only if the sequence (a_n) of m-dimensional vectors satisfies the *induced* recurrent system

$$P_l(n)a_{n+l} + \cdots + P_t(n)a_{n+t} = c_n, \tag{5}$$

where (c_n) is the sequence of coefficients of the Laurent series which is the expansion of the right-hand side $b(x)$ of the original q-difference system.
The induced system can be constructed in 3 steps:
1) rewrite the original system in the operator-matrix form $My = b$, where $M \in \mathrm{Mat}_m(\mathbb{K}[x, \sigma_q])$,
2) in the matrix M, replace $\sigma_q \to q^n, \;\; x \to \sigma^{-1}$, where σ is the shift-operator: $\sigma f_n = f_{n+1}$ for any double sided sequence (f_n),
3) rewrite the obtained system in the form (5).

Below, we will need the notion of the valuation of a series: for a non-zero Laurent series $f(x) = \sum f_i x^i$. The *valuation* of this series is

$$\mathrm{val} f(x) = \min\{i \in \mathbb{Z} \mid f_i \neq 0\},$$

and conventionally $\mathrm{val} f(x) = \infty$ for the zero series $f(x)$. The valuation of the vector whose components are series, is the minimal valuation of the components.

The *degree* of the vector with polynomial components is the maximal degree of the components. The degree of the zero polynomial is $-\infty$.

If the induced system (5) is such that $\det P_t(n)$ is a non-zero polynomial in q^n, then one can find a lower bound for valuations of formal Laurent series solutions. An upper bound for degrees can be found using the non-zero determinant of the trailing matrix. The following two theorems are from Abramov, 2002.

Theorem 1

Let recurrent system (5) *be such that if a formal Laurent series* $\sum a_n x^n$, $a_n \in \mathbb{K}^m$ *(in particular, it can be a polynomial over* $\mathbb{K}^m$*) satisfies the original q-difference system* (1), *then the sequence* (a_n) *of m-dimensional vectors satisfies* (5). *Let* $p_l(n) = \det P_l(n)$, $p_t(n) = \det P_t(n)$ *(thus,* $p_l(n), p_t(n)$ *are polynomials in* q^n*). In this case*
If the right-hand side $b(x)$ *is a Laurent series, and*

- $p_l(n)$ *is a non-zero polynomial in* q^n,
- N_l *is the set (possibly empty) of all integer roots of the equation* $p_l(n) = 0$,
- *the number* β *does not exceed the valuation of the right-hand side of the original q-difference system (*$\beta = \infty$, *when the right-hand side* $b(x)$ *is the zero column vector),*

then the valuation of any Laurent solution of system (1) *cannot be less than*

$$\min(N_l \cup \{\beta\}) + l.$$

Theorem 2

If the right-hand side $b(x)$ is polynomial, and

- $p_t(n)$ *is a non-zero polynomial in* q^n,
- N_t *is the set (possibly empty) of all integer roots of the equation* $p_t(n) = 0$,
- *the number* γ *does not exceed the degree of the right-hand side of the original q-difference system (*$\gamma = -\infty$*, when the right-hand side* $b(x)$ *is the zero column vector),*

then the degree of any polynomial solution of system (1) *cannot be bigger than*

$$\max(N_t \cup \{\gamma\}) + t.$$

If the leading or, resp., the trailing matrix of the induced system is singular then one can apply the corresponding version of the algorithm EG and with Theorems above find the needed bounds, this gives a key to construct Laurent and polynomial solutions.

Rational solutions

First, we find $U(x) \in \mathbb{K}(x)$ ($U(0) \neq 0$) and $k \in \mathbb{Z}$ such that any rational solution $y(x)$ can be written as

$$y(x) = \frac{x^k}{U(x)} z(x), \tag{6}$$

where $z(x) \in \mathbb{K}[x]^m$. Then we produce the substitution (6) for $y(x)$, and after cleaning denominators we apply an algorithm for finding polynomial solutions. The dissimilarity between x and other irreducible polynomials is such that if $p(x) \in \mathbb{K}[x]$ is irreducible and $p(0) \neq 0$, then $p(q^h x)$ is also irreducible and relatively prime with $p(x)$ for any $h \in \mathbb{Z}$, and different values of h give different irreducible polynomials. However this does not take place for the polynomial x, which is not relatively prime with qx. This gives the polynomial x a special status, which is not the case for the difference equations, when any irreducible $p(x)$ (in particular, $p(x) = x$) is relatively prime with $p(x+1)$).

We have stated that any rational solution can be considered as a Laurent solution. Thus Theorem 2 gives an opportunity to define k for the factor x^k. As for the polynomial $U(x)$, we find it in accordance with our scheme, using the "difference" algorithm with the replacement of the shift σ by the q-shift σ_q. We give some definitions and then describe the algorithm.
If $F(x)$ is a rational function then we denote by $\mathrm{den} F(x)$ the *denominator* of $F(x)$, i.e. a monic polynomial such that $F(x) = \frac{f(x)}{\mathrm{den} F(x)}$ for a polynomial $f(x)$ which is co-prime with $\mathrm{den} F(x)$. If $F(x)$ is a vector with rational function components $F_1(x), \ldots, F_m(x)$ then $\mathrm{den} F(x)$ is the least common multiple (lcm) of $\mathrm{den} F_1(x), \ldots, \mathrm{den} F_m(x)$.
We write $f(x) \perp g(x)$ for relatively prime $f(x), g(x) \in \mathbb{K}[x]$, and $f(x) \not\perp g(x)$, when the polynomials have a common divisor of positive degree.

Each polynomial $f(x) \in \mathbb{K}(x) \setminus \{0\}$ can be represented as $f(x) = x^v s(x)$, where $v \in \mathbb{Z}_{\geqslant 0}$ and the polynomial $s(x)$ is not divisible by x, i.e., $s(0) \neq 0$. In this case we will call $s(x)$ the *stem* of $f(x)$, we will use the notation $\nu(f(x))$ for v. If $\nu(f(x)) = \nu(g(x)) = 0$, then we can introduce the *q-dispersion set* of the polynomials $f(x)$ and $g(x)$:

$$\mathrm{qds}(f(x), g(x)) = \{h \in \mathbb{Z}_{\geqslant 0} \mid f(x) \not\perp g(q^h x)\}$$

and their *q-dispersion*:

$$\mathrm{qdis}(f(x), g(x)) = \max(\mathrm{qds}(f(x), g(x)) \cup \{-\infty\}).$$

Similarly to the difference case, the q-dispersion is either a non-negative integer, or is equal to $-\infty$, the latter takes place if and only if $f(x) \perp g(q^h x)$ for all $h \in \mathbb{Z}_{\geqslant 0}$.

As we have already said, if a polynomial $p(x) \in \mathbb{K}[x]$ is irreducible and $\nu(p(x)) = 0$, then the polynomial $p(q^h x)$, $h \in \mathbb{Z}_{\geqslant 0}$, is also irreducible, and such polynomials are relatively prime for different values of h. This implies that if $\nu(f(x)) = \nu(g(x)) = 0$, then $\mathrm{qds}(f(x), g(x))$ is a finite set. This set can be found, e.g., by computing all the roots having the form $\lambda = q^h$, $h \in \mathbb{Z}_{\geqslant 0}$, of the equation $R(\lambda) = 0$, where $R(\lambda) = \mathrm{Res}_x(f(x), g(\lambda x))$, or by an analog of the algorithm of Y.Man & F.Wright, 1994, which is originally for the difference case. (In Abramov& Bronstein, 2002, an algorithm was proposed which is applicable also in the case where q is algebraic number which is not a root of 1.)

Thus, when k is found, we have to construct such a polynomial $U(x)$ that possess the following properties

(a) $\nu(U(x)) = 0$,

(b) if the original system has a rational solution having the denominator $u(x)$, then $U(x)$ is divisible by the stem of $u(x)$.

When k and $U(x)$ are known we can use substitution (6).
One can find $U(x)$ similarly to the universal denominator in the difference case (Abramov, 2012). In the algorithm, we use the notation $\gcd(f(x), g(x))$ for the greatest common divisor of polynomials $f(x), g(x)$.

$A(x) = (\det \bar{A}_r(q^{-r}x))/x^{a_r}, \; B(x) = (\det \bar{\bar{A}}_0(x))/x^{a_0}$,
where
$a_r = \nu\,(\det \bar{A}_r(x)), \;\; a_0 = \nu\,(\det \bar{\bar{A}}_0(x))$.
Compute $H = \mathrm{qds}(A(x), B(x))$. If $H = \varnothing$ then stop with $U(x) = 1$ (in the sequel we suppose that $H = \{h_1, h_2, \ldots, h_s\}$ with $h_1 > h_2 > \cdots > h_s$, $s \geqslant 1$).
Set $U(x) = 1$.
for i=1 to s do
$N(x) = \gcd(A(x), B(q^{h_i}x))$
$A(x) = A(x)/N(x)$
$B(x) = B(x)/N(q^{-h_i}x)$
$U(x) = U(x)\prod_{j=0}^{h_i} N(q^{-j}x)$.
od.
Return $U(x)$ and stop.

Theorem 3

Let each solution of the original q-difference system of the form (1) *be also a solution of systems* (3), (4), *and* $\det \bar{A}_r(x)$, $\det \bar{\bar{A}}_0(x)$ *be non-zero. Then the polynomial* $U(x)$ *computed by the latter algorithm possesses properties (a), (b) formulated above.*

The main idea of the proof is similar to the idea used for the difference case. First of all, if $f(x), p(x)$ are polynomials and $p(x)$ is irreducible then the valuation $\mathrm{val}_{p(x)} f(x)$ is defined to be the greatest $n \in \mathbb{Z}_{\geqslant 0}$ such that $f(x)$ is divisible by $p(x)^n$ ($\mathrm{val}_{p(x)} 0 = \infty$, and $\mathrm{val}_{p(x)} \frac{f(x)}{g(x)} = \mathrm{val}_{p(x)} f(x) - \mathrm{val}_{p(x)} g(x)$).
The valuation of a vector whose components are polynomials or rational functions is the minimal of the component valuations.

The following statement can be proven: For any rational solution $y(x)$ of (1) and any irreducible $p(x)$ we have

$$\mathrm{val}_{p(x)} y(x) \geqslant \max \left\{ -\sum_{n \in \mathbb{Z}_{\geqslant 0}} \mathrm{val}_{p(q^n x)} A(x),\ -\sum_{n \in \mathbb{Z}_{\geqslant 0}} \mathrm{val}_{p(q^{-n} x)} B(x) \right\}, \quad (7)$$

and $\mathrm{val}_{p(x)} U(x)$ does not exceed the valuation of the right-hand side of (7).

Remark that similarly to the difference case (Abramov, Gheffar, Khmelnov, 2010), inequality (7) can be taken as a base for another algorithm for constructing the polynomial $U(x)$. That algorithm uses the full factorization of $A(x), B(x)$ which is used also for the dispersion computation by the q-version of the Man & Wright algorithm. However the algorithm given above is more convenient for implementation.