Search for Hypergeometric Solutions of q-Difference Systems by Resolving Sequences

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We consider a homogeneous linear q-difference system

$$A_r(x)y(q^r x) + \cdots + A_1(x)y(q\,x) + A_0(x)y(x) = 0,$$

where

- $A_r(x), \ldots, A_1(x) \in \mathrm{Mat}_m(K(x, q))$,
- $A_r(x) \not\equiv 0, A_0(x) \not\equiv 0$,
- $m \in \mathbb{Z}_{\geqslant 1}$,
- K is an algebraically closed field of characteristic 0,
- q is transcendental over K,
- $y(x) = (y_1(x), \ldots, y_m(x))^T$ is a vector-column of unknown functions,
- $x = q^n,\ n \in \mathbb{Z}_{\geqslant 0}$.

For example (Maple 2017)

$$> S1 := \begin{bmatrix} x & q x^2 \\ 2 q x + x^2 & q^2 x^2 + x^3 q \end{bmatrix} \cdot y(q^2 x) + \begin{bmatrix} -q^3 x^2 + q^2 - q x & -q^3 x^3 + q^2 x - q x^2 \\ -q^4 x^2 - q^3 x^3 + q^3 - q^2 x & -q^4 x^3 - q^3 x^4 + q^3 x - x^3 q \end{bmatrix} \cdot y(q x) + \begin{bmatrix} -q^4 x + q^3 x^2 & -q^3 x^2 + q^2 x^3 \\ -q^5 x + q^3 x^3 - q^2 x^2 & -q^4 x^2 + q^2 x^4 \end{bmatrix} \cdot y(x) = 0 :$$

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$$> S2 := \{ (-q^5 x + x^3 q^3 - q^2 x^2) y_1(x) + (-q^4 x^2 - x^3 q^3 + q^3 - q^2 x) y_1(q x) + (2 q x + x^2) y_1(q^2 x) + (-q^4 x^2 + q^2 x^4) y_2(x) + (-q^4 x^3 - q^3 x^4 + q^3 x - x^3 q) y_2(q x) + (q^2 x^2 + x^3 q) y_2(q^2 x) = 0, (-q^4 x + q^3 x^2) y_1(x) + (-q^3 x^2 + q^2 - q x) y_1(q x) + y_1(q^2 x) x + (-q^3 x^2 + q^2 x^3) y_2(x) + (-x^3 q^3 + q^2 x - q x^2) y_2(q x) + y_2(q^2 x) q x^2 = 0 \} :$$

\>

$$> \langle y_1(x), y_2(x) \rangle = \langle LinearFunctionalSystems{:-}PolynomialSolution(S2, [y_1(x), y_2(x)]) \rangle;$$

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \_c_1 x \\ -\_c_1 q \end{bmatrix} \qquad \textbf{(1)}$$

\>

– S. Abramov, P. Paule, M. Petkovšek. q-Hypergeometric solutions of q-difference equations, Discrete Mathematics 180, 1998, pp. 3–22.

$h(x)$ is called a q-hypergeometric term over K if

$$\frac{h(q\,x)}{h(x)} \in K(x,q) \quad \left(\frac{h(q^{n+1})}{h(q^n)} \in K(q^n,q) \right).$$

For example,

$$h(x) = \frac{1}{x}: \quad \frac{h(q\,x)}{h(x)} = \frac{1}{q}$$

$$h(x) = h(q^n) = q^{\binom{n}{2}}: \quad \frac{h(q^{n+1})}{h(q^n)} = q^n = x$$

$$h(x) = h(q^n) = h(q^{n_0}) \prod_{i=n_0}^{n-1} r(q^i): \quad \frac{h(q^{n+1})}{h(q^n)} = r(q^n) = r(x)$$

Denote by $\mathcal{L}_{H_K}$ the linear space of finite linear combinations of q-hypergeometric terms over K with coefficients in $K(q)$.
For example: $q^n + (-1)^n \in \mathcal{L}_{H_K}$.

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A basis of solutions belonging to $\mathcal{L}^m_{H_K}$ of

$$A_r(x)y(q^r x) + \cdots + A_1(x)y(q\,x) + A_0(x)y(x) = 0$$

consists of elements of the form

$$h(x)R(x),$$

where $h(x)$ is a q-hypergeometric term and $R(x) \in K(x, q)^m$.

We propose an algorithm to find such basis.

– S. Abramov, M. Petkovšek, A. Ryabenko. Hypergeometric solutions of first-order linear difference systems with rational-function coefficients, 18-th worshop on computer algebra, Dubna 2015, and CASC'2015, LNCS 9301, 2015, pp. 1–14.

– S. Abramov, M. Petkovšek, A. Ryabenko. Resolving sequences of operators for linear ordinary differential and difference systems, Computational Mathematics and Mathematical Physics 56, 2016, pp. 894–910.

For differential systems with rational-function coefficients to find all formal exponential-logarithmic solutions and for difference systems with rational-function coefficients to find all hypergeometric solutions, the notion *resolving sequence of operators (equations)* was introduced.

1. Compute a resolving sequence of equations for the given system.
2. Find bases of solutions belonging to $\mathcal{L}_{H_K}$ for all equations from the resolving sequence: $h_1(x), \ldots, h_{s_1}(x)$.
3. Let $\tilde{h}_1(x), \ldots, \tilde{h}_{s_2}(x)$ be all non-similar q-hypergeometric terms from $h_1(x), \ldots, h_{s_1}(x)$: $\tilde{h}_i(x)/\tilde{h}_j(x) \notin K(x,q), \quad i \neq j$.
4. For each $\tilde{h}_j(x)$:
    4.1 Substitute $y(x) = \tilde{h}_j(x)R(x)$ into the given system, where $R(x)$ is a vector-column of new unknown functions.
    4.2 Find a basis of solutions belonging to $K(x,q)$ for the new system: $R_{j,1}(x), \ldots, R_{j,t_j}(x)$.
5. The set of all solutions

$$\tilde{h}_1(x)R_{1,1}(x), \ldots, \tilde{h}_1(x)R_{1,t_1}(x),$$
$$\ldots,$$
$$\tilde{h}_{s_2}(x)R_{s_2,1}(x), \ldots, \tilde{h}_{s_2}(x)R_{s_2,t_{s_2}}(x)$$

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1. Resolving sequences

For the q-difference system

$$A_r(x)y(q^r x) + \cdots + A_1(x)y(q\,x) + A_0(x)y(x) = 0,$$

where $y(x) = (y_1(x), \ldots, y_m(x))^T$, with invertible leading and trailing matrices: $\det A_r(x) \not\equiv 0$, $\det A_0(x) \not\equiv 0$, a *resolving sequence* of equations is

$$L_1 y_{\ell_1}(x) = 0, \ldots,\ L_j y_{\ell_j}(x) = 0, \ldots,\ L_p y_{\ell_p}(x) = 0,$$

where

- $L_1, \ldots, L_p \in K[x, q][Q]$,
- $Q\,y(x) = y(q\,x)$,
- $\ell_1, \ldots, \ell_p \in \{1, 2, \ldots, m\}$, $\ell_i \neq \ell_j$ for $i \neq j$.

1. Resolving sequences

Proposition. Let

- $L_1 y_{\ell_1}(x) = 0, \ldots, L_p y_{\ell_p}(x) = 0$ be a resolving sequence for the system S;
- $y(x) = h(x)\,R(x)$ be a non-zero solution of S, where $h(x)$ be a q-hypergeometric term and $R(x) \in K(x,q)^m$.

Then there exists j, $1 \leqslant j \leqslant p$, such that $L_j y_{\ell_j}(x) = 0$ has a non-zero solution of the form $h(x)f(x)$, $f(x) \in K(x,q)$.

```
> RS1 := LqRS:-ResolvingSequence(S1, y(x), select_indicator = 1);
```

$$RS1 := \left[-q\,x\,y_1(x) + (-q + x)\,y_1(q\,x) + y_1(q^2 x),\ (-q^2 x + q\,x^2)\,y_2(x) + (-q^2 x^2 + q - x)\,y_2(q\,x) + x\,y_2(q^2 x)\right] \qquad (2)$$

```
> RS2 := LqRS:-ResolvingSequence(S1, y(x), select_indicator = 2);
```

$$\begin{aligned} RS2 := \Big[& \big(-q^{10} x^6 + q^9 x^7 + q^9 x^6 - q^8 x^7 - 2 q^9 x^5 + 2 q^8 x^6 + q^9 x^4 - 2 q^7 x^6 + q^6 x^7 - q^8 x^4 + 2 q^6 x^6 - q^5 x^7 + q^7 x^3 \\ & - q^6 x^4 + 2 q^4 x^3 - 2 q^3 x^4 - q^4 x^2 + q^3 x^3 + q^2 x^3 - q\,x^4 - q^2 x^2 + x^3 q\big)\,y_2(x) + \big(-q^{10} x^7 - q^9 x^6 + q^8 x^7 \\ & - q^{10} x^4 + 4 q^9 x^5 - 3 q^8 x^6 - q^7 x^7 + q^{10} x^3 - q^9 x^4 - q^8 x^5 + q^7 x^6 - q^9 x^3 + 2 q^8 x^4 - q^7 x^5 + q^5 x^7 - q^8 x^3 \\ & + 2 q^7 x^4 + q^6 x^5 - 2 q^5 x^6 + q^8 x^2 - 2 q^7 x^3 + 2 q^6 x^4 - q^5 x^5 + q^4 x^6 - q^6 x^3 + q^4 x^5 + q^4 x^4 - q^3 x^5 + q^5 x^2 \\ & - 3 q^4 x^3 + 2 q^3 x^4 - q^5 x + 2 q^4 x^2 - 2 q^3 x^3 + q^2 x^4 - q\,x^2 + x^3 + q\,x - x^2\big)\,y_2(q\,x) + \big(q^{10} x^7 - q^9 x^7 + q^{11} x^4 \\ & - 2 q^{10} x^5 + q^9 x^6 - q^9 x^5 + q^8 x^6 + q^7 x^7 + q^7 x^6 - q^6 x^7 + q^9 x^3 - q^8 x^4 + q^7 x^5 - 2 q^6 x^6 + 2 q^8 x^3 - 4 q^7 x^4 \\ & + q^6 x^5 + 2 q^5 x^6 - 2 q^8 x^2 + 2 q^7 x^3 + 2 q^6 x^3 - q^5 x^4 - q^3 x^6 - q^6 x^2 + q^3 x^5 - q^3 x^4 + q^3 x^3 - q^2 x^4 - q^2 x^3 \\ & + q\,x^4 - q^3 x + 3 q^2 x^2 - 2 x^3 q + q^3 - 2 q^2 x + 2 q\,x^2 - x^3 - 2 q\,x + 2 x^2 + q - x\big)\,y_2(q^2 x) + \big(q^{10} x^6 - q^{11} x^4 \\ & + 2 q^{10} x^5 - q^9 x^6 + q^{10} x^4 - q^9 x^5 - q^{10} x^3 + q^8 x^5 - q^8 x^3 + 2 q^7 x^4 - q^6 x^5 + q^8 x^2 - 2 q^7 x^3 + q^6 x^4 - q^4 x^6 \\ & - q^4 x^5 + q^3 x^6 + q^3 x^5 - q^5 x^2 + q^4 x^3 + q^3 x^4 - q^2 x^5 + q^5 x - q^4 x^2 + q^4 x - 2 q^3 x^2 + q^3 x - 2 q^2 x^2 + x^3 q \\ & - q^3 + 3 q^2 x - 2 q\,x^2 + q\,x - x^2 - q + x\big)\,y_2(q^3 x) + \big(-q^6 x^5 + q^7 x^3 - 2 q^6 x^4 + q^5 x^5 - q^6 x^3 + q^5 x^4 + q^6 x^2 \\ & - q^4 x^4 - q^3 x^5 + q^2 x^5 - q^4 x + 2 q^3 x^2 - q^2 x + q\,x^2\big)\,y_2(q^4 x)\Big] \end{aligned} \qquad (3)$$

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>
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2. Bases of hypergeometric solutions for all equations from the resolving sequence

– S. Abramov, P. Paule, M. Petkovšek. q-Hypergeometric solutions of q-difference equations, Discrete Mathematics 180, 1998, pp. 3–22.

```
> st := time( ):
  QDifferenceEquations:-QHypergeometricSolution(RS1[1], y_1(x), output = Certificate);
  time( ) - st;
```

$$\{q\}$$

$$0.123 \qquad (4)$$

```
> st := time( ):
  QDifferenceEquations:-QHypergeometricSolution(RS1[2], y_2(x), output = Certificate);
  time( ) - st;
```

$$\{q\,x\}$$

$$0.080 \qquad (5)$$

```
> st := time( ):
  QDifferenceEquations:-QHypergeometricSolution(RS2[1], y_2(x), output = Certificate);
  time( ) - st;
```

$$\{1, q\,x\}$$

$$5.857 \qquad (6)$$

```
>
```

A normal form for rational functions

[APP, 1998] Theorem 1.

$$r(x) = z\,\frac{a(x)}{b(x)}\,\frac{c(q\,x)}{c(x)}\,\frac{d(x)}{d(q\,x)} = z\,U(x)\,\frac{V(q\,x)}{V(x)},$$

where

- $z \in K(q)$;
- $a(x), b(x), c(x), d(x) \in K(q)[x]$ are monic polynomials in x;
- $a(x) \perp b(q^n\,x)$ for $n \in \mathbb{Z}$;
- $a(x) \perp c(x)\,d(q\,x)$; $b(x) \perp c(q\,x)\,d(x)$;
- $c(0) \neq 0$, $d(0) \neq 0$;
- $U(x) = \frac{a(x)}{b(x)}$, $V(x) = \frac{c(x)}{d(x)}$.

3. Similar hypergeometric terms

Let a set of hypergeometric terms $h_1(x), \dots, h_{s_1}(x)$ be given in the previous step. All terms are presented by their certificates $r_1(x), \dots, r_{s_1}(x)$. For $j = 1, \dots, s_1$,

$$h_j(q^k) = h_j(q^{n_0}) \prod_{i=n_0}^{n-1} r_j(q^i) = C_j\, z_j^n V_j(q^n) \prod_{i=n_0}^{n-1} U_j(q^i).$$

For $i \neq j$,

$$h_i(x) \sim h_j(x) \iff U_i(x) = U_j(x) \text{ and } \frac{z_i}{z_j} = q^k \ \ (k \in \mathbb{Z}).$$

$$h_j(x) \in K(x, q) \iff U_j(x) = 1 \text{ and } z_j = q^k \ \ (k \in \mathbb{Z}).$$

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$$h_j(q^k) = h_j(q^{n_0}) \prod_{i=n_0}^{n-1} r_j(q^i) = C_j\, z_j^n V_j(q^n) \prod_{i=n_0}^{n-1} U_j(q^i).$$

For $i \neq j$,

$$h_i(x) \sim h_j(x) \Longleftrightarrow U_i(x) = U_j(x) \text{ and } \frac{z_i}{z_j} = q^k \ \ (k \in \mathbb{Z}).$$

$$h_j(x) \in K(x, q) \Longleftrightarrow U_j(x) = 1 \text{ and } z_j = q^k \ \ (k \in \mathbb{Z}).$$

3. Similar hypergeometric terms

Let a set of hypergeometric terms $h_1(x), \ldots, h_{s_1}(x)$ be given in the previous step. All terms are presented by their certificates $r_1(x), \ldots, r_{s_1}(x)$. For $j = 1, \ldots, s_1$,

$$h_j(q^k) = h_j(q^{n_0}) \prod_{i=n_0}^{n-1} r_j(q^i) = C_j\, z_j^n V_j(q^n) \prod_{i=n_0}^{n-1} U_j(q^i).$$

For $i \neq j$,

$$h_i(x) \sim h_j(x) \Longleftrightarrow U_i(x) = U_j(x) \text{ and } \frac{z_i}{z_j} = q^k \ \ (k \in \mathbb{Z}).$$

$$h_j(x) \in K(x,q) \Longleftrightarrow U_j(x) = 1 \text{ and } z_j = q^k \ \ (k \in \mathbb{Z}).$$

4.1 The substitution $y(x) = h(x)\,R(x)$ into the system

For $h(x)$ with the certificate $r(x) = z\,q^k\,U(x)\,\frac{V(q\,x)}{V(x)}$, we substitute

$$y(x) = z^n \left(\prod_{i=n_0}^{n-1} U(q^i)\right) R(x),$$

where $R(x)$ is a vector-column of new unknown functions, into the system

$$A_r(x)y(q^r x) + \cdots + A_1(x)y(q\,x) + A_0(x)y(x) = 0,$$

$$\Downarrow$$

$$B_r(x)R(q^r x) + \cdots + B_1(x)R(q\,x) + B_0 R(x) = 0,$$

where for $j = 0, 1, \ldots, r$

$$B_j(x) = z^j\,U(q^{j-1}x) \cdots U(q\,x)\,U(x)A_j(x).$$

4.2 Rational solutions for q-difference systems

- S. Abramov. EG-eliminations as a tool for computing rational solutions of linear q-difference systems of arbitrary order with polynomial coefficient, Computer algebra: International Conference Materials. Moscow, October 30 – November 3, 2017, pp.54–60.

The algorithm has been implemented in Maple 2017 as the procedure **HypergeometricSolution** in a package **LqRS** (Linear q-Recurrence Systems) (available on *http://www.ccas.ru/ca/lqrs*).

```
> st := time( ) :
  LqRS:-HypergeometrisSolution(S1, y(x), n, 'output'='basis');
  time( ) − st;
```

$$\left[\begin{bmatrix} q^n \\ -q \end{bmatrix}, \begin{bmatrix} 0 \\ q^{\binom{n}{2}} q^n \end{bmatrix}\right]$$

$$0.916 \qquad (7)$$

```
>
> LqRS:-RationalSolution(S1, y(x), k)
```

$$\begin{bmatrix} x\,\_c_1 \\ -\_c_1\, q \end{bmatrix} \qquad (8)$$

```
>
```