Pergamon

Comput. Maths. Math. Phys., Vol.35, No.5, pp. 539–551, 1995

0965-5425(95)00074-7

THE CONVERGENCE OF PROXIMAL METHODS TO FIXED POINTS OF EXTREMAL MAPPINGS AND ESTIMATES OF THEIR RATE OF CONVERGENCE[1]

A.S. ANTIPIN
Moscow
(Revised version 16 December 2002)

It is proved that proximal methods converge to the sharp, strongly convex, and degenerate fixed points of extremal mappings.

1. STATEMENT OF THE PROBLEM

We will consider the problem of calculating the fixed point of the extremal mapping

$$v^* \in \mathrm{Argmin}\{\Phi(w, v^*) : w \in \Omega\}, \quad v \in \Omega, \tag{1.1}$$

where the function $\Phi(w, v)$ is defined on the square $\Omega \times \Omega$ and $\Omega \in \mathbb{R}^n$. We shall assume that $\Phi(w, v)$ is convex with respect to the variable w for each fixed v. Also, the set $\mathrm{Argmin}\{\Phi(w, v) : w \in \Omega\}$, which depends on v, is non-empty for all v and a solution of problem (1.1) always exists.

Many well-known mathematical models can be reduced to the structure (1.1), including, primarily, inverse optimization problems [1], n-person zero-sum games (saddle problems) and non-zero sum games, variational inequalities, problems of economic equilibrium [2], problems of the distribution of resources at unstable prices [3], ill-posed problems [4] and multicriterion decision-making in conditions of indeterminacy [5], etc. The importance and urgency of developing methods for calculating fixed points are obvious from this list.

We know that the convergence of any process and estimates of the rate of convergence depend on the behaviour of the objective function in the neighborhood of the solution of the problem. This is reflected in optimization by the concepts of "sharp", "strongly convex" and "degenerate" minima. Following those ideas, we will introduce a parametric scale for the types of equilibrium using the parametric family of inequalities

$$\Phi(w, w) - \Phi(v^*, w) \geq \gamma |w - v^*|^\sigma, \tag{1.2}$$

which hold for all $w \in S = \{w : |w - v^*| < 1, \ w \in \Omega\}$, and for the unique solution $v^* \in \Omega$ of problem (1.1), where $\gamma > 0$ is a constant and the parameter $0 < \sigma \leq \infty$. As $\sigma \to \infty$, the

[1] *Zh. Vychisl. Mat. Mat. Fiz.*, Vol.35, No.5, pp. 688–704, 1995
Financial support for this research was provided by the Russian Foundation for Basic Research (Project No. 94-01-00005).

quantity $|w - v^*|^\sigma \to |w - v^*|^\infty \equiv 0$ for all $w \in S$. The unit sphere may be replaced by any other sphere of smaller radius.

In the case where all the solutions of the problem form a closed set $\Omega^* \in \Omega$, inequality (1.2) takes the form

$$\Phi(w, w) - \Phi(\pi_{\Omega^*}(w), w) \geq \gamma |w - \pi_{\Omega^*}(w)|^\sigma, \tag{1.3}$$

which holds for all $w \in \{w : |w - \pi_{\Omega^*}(w)| < 1,\ w \in \Omega\}$, $\gamma > 0$ is a constant, $\sigma > 0$ is a parameter, and $\pi_{\Omega^*}(w)$ is the operator of projection of the vector w on to the set Ω^*. If the set Ω^* consists of one point, (1.3) becomes (1.2).

There are two other inequalities, closely connected with (1.2) and (1.3), which have a more symmetric form with respect to their variables. The first describes the behaviour of the objective function of problem (1.1) in the neighborhood of the unique solution:

$$[\Phi(w, w) - \Phi(v^*, w)] - [\Phi(w, v^*) - \Phi(v^*, v^*)] \geq \gamma |w - v^*|^\sigma \tag{1.4}$$

for all $w \in \{w : |w - v^*| < 1,\ w \in \Omega\}$, $\gamma > 0$ is a constant, and $\sigma > 0$ is a parameter. The second inequality describes the same situation, but for the case when the set of solutions of problem (1.1) is closed:

$$[\Phi(w, w) - \Phi(\pi_{\Omega^*}(w), w)] - [\Phi(w, \pi_{\Omega^*}(w)) - \Phi(\pi_{\Omega^*}(w), \pi_{\Omega^*}(w))] \geq \gamma |w - \pi_{\Omega^*}(w)|^\sigma. \tag{1.5}$$

This inequality is true for all $w \in \{w : |w - \pi_{\Omega^*}(w)| \leq 1,\ w \in \Omega\}$, $\gamma > 0$ is a constant, and $\sigma > 0$ is a parameter. If the set Ω^* consists of one point, inequality (1.5) becomes (1.4).

Since, according to (1.1), $\Phi(w, v^*) - \Phi(v^*, v^*) \geq 0$ or $\Phi(w, \pi_{\Omega^*}(w)) - \Phi(\pi_{\Omega^*}(w), \pi_{\Omega^*}(w)) \geq 0$, (1.2) and (1.3) can be obtained directly from (1.4) and (1.5). One or other inequalities will be used to an equal extent below.

If the function $\Phi(w, v)$ is differentiable and convex with respect to the first variable, inequalities (1.5) and (1.4) can be represented in the equivalent forms:

$$\langle \nabla_1 \Phi(w, w) - \nabla_1 \Phi(\pi_{\Omega^*}(w), \pi_{\Omega^*}(w)), w - \pi_{\Omega^*}(w) \rangle \geq \gamma |w - \pi_{\Omega^*}(w)|^\sigma$$

and, correspondingly,

$$\langle \nabla_1 \Phi(w, w) - \nabla_1 \Phi(w^*, v^*), w - v^* \rangle \geq \gamma |w - v^*|^\sigma \qquad \forall\, w \in \Omega.$$

If the objective function in (1.1) is a bilinear function $\Phi(w, v) = \langle \Phi w, v \rangle$ with a square matrix Φ, of the same dimensions as the vector w, inequality (1.4) will become:

$$\begin{aligned} & (\langle \Phi w, w \rangle - \langle \Phi v^*, w \rangle) - (\langle \Phi w, v^* \rangle - \langle \Phi v^*, v^* \rangle) = \\ = \ & \langle \Phi(w - v^*), w \rangle - \langle \Phi(w - v^*), v^* \rangle = \\ = \ & \langle \Phi(w - v^*), w - v^* \rangle \geq \gamma |w - v^*|^\sigma \qquad \forall w \in \Omega. \end{aligned}$$

When $\sigma = 2$, the inequality takes the form of the strong condition for the symmetric matrix Φ to be positive definite:

$$\langle \Phi(w - v^*), w - v^* \rangle \geq \gamma |w - v^*|^2 \qquad \forall w \in \Omega,$$

and γ is the smallest eigenvalue. When $\sigma = \infty$, we obtain the condition for Φ to be non-negative:

$$\langle \Phi(w - v^*), w - v^* \rangle \geq 0 \qquad \forall w \in \Omega.$$

If the objective function of the original problem (1.1) depends on only the one variable w, (1.1) becomes the optimization problem:

$$v^* \in \text{Argmin}\,\{\Phi(w) : w \in \Omega\}, \tag{1.6}$$

and the parametric family of inequalities (1.2) takes the form

$$\Phi(w) - \Phi(v^*) \geq \gamma|w - v^*|^\sigma \qquad \forall w \in \Omega, \tag{1.7}$$

$\gamma > 0$ is a constant and $\sigma > 0$ is a parameter. If the parameter σ in (1.7) successively takes the values 1, 2 and ∞, the minimum of problem (1.6) will become, respectively, either a sharp minimum [6], or strongly convex [6, 7] or degenerate.

We now turn to inequality (1.2). Here each value of the parameter σ corresponds to one kind of equilibrium: it will be called a sharp equilibrium when $\sigma = 1$, a strongly convex equilibrium when $\sigma = 2$, an exponential equilibrium when $\sigma > 2$, and a degenerate equilibrium when $\sigma = \infty$. As the parameter σ varies continuously in the interval $0 < \sigma \leq \infty$, the type of equilibrium also changes continuously, corresponding to the continuous change in behaviour of the objective function in the neighborhood of the solution of the initial problem. A question that arises is how gradient and proximal methods, for example, which should converge to these equilibria, behave. Their behaviour should obviously adjust to the behaviour of the objective function in the neighborhood of equilibrium. In the general case, the picture is as follows [8]: for parameter values $\sigma < 2$, the methods converge after a finite number of iterations, when $\sigma = 2$ the bounds for the rate of convergence of the processes are exponential or the rate is that of a geometric progression, when $\sigma > 2$ the bounds have a power dependence (negative power) and, finally, when $\sigma = \infty$, the bounds degenerate and the method may converge as slowly as desired. Our purpose here is to describe how the proximal method behaves for solving problem (1.1) with values of the parameter $\sigma = 1, 2$ and ∞.

2. EQUILIBRIUM PROBLEMS

In this section we will consider specific problems of the type (1.1), the solutions of which satisfy inequality (1.3) or versions thereof for different values of the parameter σ.

1. *Saddle problems*

Consider the problem of calculating the saddle point of a convex-concave degenerate function, that is, a point (x^*, p^*) which is a solution of the system of inequalities

$$L(x^*, p) \leq L(x^*, p^*) \leq L(x, p^*) \qquad \forall x \in Q \subseteq \mathbb{R}^n, \quad p \in P \subseteq \mathbb{R}^m,$$

where the function $L(x, p)$ is convex with respect to x and concave with respect to p. The function $L(x, p)$ is not strongly convex with respect to one variable and strongly concave with respect to the other. Nor is it assumed to be differentiable. The sets Q and P are convex and closed.

In the general case, the problem of finding a saddle point of the function $L(x, y)$ can always be reformulated as a two-person zero-sum game. A solution of the game (or its equilibrium point, according to Nash) is a point (x^*, p^*) which satisfies the system of extremal inclusions:

$$x^* \in \text{argmin}\,\{L(z, p^*) : \; z \in Q\}, \qquad p^* \in \text{argmin}\,\{-L(x^*, y) : \; y \in P\}. \tag{2.1}$$

It is most convenient to associate the game (2.1) with a normalized function of the form [9]

$$\Phi(w, v) = L(z, p) - L(x, y), \tag{2.2}$$

where $w = (z, y)$, $v = (x, p)$. This function is also defined in the space of variables z, y, x, p, that is, in a space of twice as many dimensions as the original. Since the function $\Phi(w, v)$ is separable with respect to the variables z and y and the set $\Omega = Q \times P$ has a modular structure, the problem

$$v^* \in \operatorname{argmin}\{\Phi(w, v^*) : \ w \in \Omega\}, \quad v \in \Omega, \tag{2.3}$$

is equivalent to problem (2.1), and the sets of solutions of the problems are the same. If the range of variation of the variables w and v is not the direct product $\Omega \times \Omega$, the set of normalized solutions of problem (2.3) contains only some of the solutions of problem (2.1).

We will check that the function (2.2) satisfies condition (1.5). With that aim, we verify the following properties [10, 11]:

$$\Phi(v, v) = 0 \qquad \forall\, v \in \Omega, \tag{2.4}$$

$$\Phi(w, v^*) + \Phi(v^*, w) = 0 \quad \forall\, w \in \Omega, \tag{2.5}$$

The first property means that the function $\Phi(w, v)$ is equal to zero on the diagonal of the square, that is, when $w = v$. This is the reason for the appellation "n-person zero-sum games". It is obvious that (2.4) is satisfied by the function $\Phi(w, v) = L(z, p) - L(x, y)$, where $w = (z, y)$, $v = (x, p)$, since when $w = v$, we have $\Phi(v, v) = L(x, p) - L(x, p) = 0$.

It is also easy to see that the second property is true. Let $v = v^*$. Then $\Phi(w, v^*) = L(z, p^*) - L(x^*, y)$. Since the ranges of variation of the variables $w \in \Omega$ and $v \in \Omega$ are the same, we put $w = v^*$ and $v = w$ in $\Phi(w, v)$; then $\Phi(v^*, w) = L(x^*, y) - L(z, p^*)$. Thus $\Phi(w, v^*) + \Phi(v^*, w) = L(z, p^*) - L(x^*, y) + L(x^*, y) - L(z, p^*) = 0$. The argument applies if the domain of definition of the function $\Phi(w, v)$ is the square $\Omega \times \Omega$. However, it remains valid if the domain of definition of the function is a convex closed set that is symmetrically positioned in relation to the diagonal of the square ($w = v$), or in other words, a set which contains the point (v_0, w_0) as well as the point (w_0, v_0).

Properties (2.4), (2.5) combined ensure that inequality (1.4) or (1.5) is satisfied when $\sigma = \infty$:

$$\Phi(w, w) - \Phi(v^*, w) - \Phi(w, v^*) + \Phi(v^*, v^*) = 0 \quad \forall\, w \in \Omega, \ \forall v^* \in \Omega^*. \tag{2.6}$$

2. *n-person non-zero sum games*

We now consider the two-person game:

$$x^* \in \operatorname{argmin}\{f(z) + L(z, p^*) : \ z \in Q\}, \qquad p^* \in \operatorname{argmin}\{\varphi(y) - L(x^*, y) : \ y \in P\}, \tag{2.7}$$

where the functions $f(z)$, $\varphi(y)$ are convex with respect to their variables, and $L(z, p)$ is a saddle function. Assume that a solution of problem (2.7) exists. The most typical example of the structure (2.7) is that of quadratic problems, such as

$$x^* \in \operatorname{argmin}\{0.5(z-1)^2 + zp^* : \ z \in \mathbb{R}^1\}, \qquad p^* \in \operatorname{argmin}\{0.5(y-3)^2 - x^* y : \ y \in \mathbb{R}^1\}.$$

The fixed point (x^*, p^*) here has coordinates $(-1, 2)$.

Problem (2.7) is not an antagonistic game and thus its solution, Nash equilibrium, reflects the notion of compromise. We write out the normalized function for this problem and verify that it is a game with a non-zero sum:

$$\Phi(w, v) = f(w_1) + L(w_1, v_2) - L(v_1, w_2) + \varphi(w_2). \tag{2.8}$$

In fact, when $w = v$ from (2.8) we have $\Phi(w, w) = f(w_1) + L(w_1, w_2) - L(w_1, w_2) + \varphi(w_2) = f(w_1) + \varphi(w_2) \neq 0$ for all w_1 and w_2 of Ω.

We will now check that inequality (1.4) holds for problem (2.7) when $\sigma = \infty$. From (2.8) we have

$$\begin{aligned} \Phi(w,w) &- \Phi(v^*,w) - \Phi(w,v^*) + \Phi(v^*,v^*) = \\ &= f(w_1) + L(w_1,w_2) - L(w_1,w_2) + \varphi(w_2) - \\ &- f(v_1^*) - L(v_1^*,w_2) + L(w_1,v_2^*) - \varphi(v_2^*) - \\ &- f(w_1) - L(w_1,v_2^*) + L(v_1^*,w_2) - \varphi(w_2) + \\ &+ f(v_1^*) + L(v_1^*,v_2^*) - L(v_1^*,v_2^*) + \varphi(v_2^*) = 0 \end{aligned}$$

for all w_1 and w_2 from Ω. Thus, when $\sigma = \infty$ inequality (1.4) is satisfied for the non-zero sum game (2.7).

3. *Strongly convex equilibrium*

In problem (1.1), let the objective function have the form $\Phi(w,w) = 0.5|Aw + Cv|^2 = 0.5|Aw|^2 + \langle Aw, Cv\rangle + 0.5|Cv|^2$. We will check that inequalities (1.4) are satisfied for $\sigma = 2$:

$$\begin{aligned} &\Phi(w,w) - \Phi(v^*,w) - \Phi(w,v^*) + \Phi(v^*,v^*) = \\ = \;& 0.5|Aw|^2 + \langle Aw, Cw\rangle + 0.5|Cw|^2 - 0.5|Av^*|^2 - \langle Av^*, Cw\rangle - \\ - \;& 0.5|Cw|^2 - 0.5|Aw|^2 - \langle Aw, Cv^*\rangle - 0.5|Cv^*|^2 + 0.5|Av^*|^2 + \\ + \;& \langle Av^*, Cv^*\rangle + 0.5|Cv^*|^2 = \langle A(w-v^*), Cw\rangle - \langle A(w-v^*), Cv^*\rangle = \\ = \;& \langle A(w-v^*), C(w-v^*)\rangle = \langle C^\top A(w-v^*), w-v^*\rangle \ge \gamma|w-v^*|^2. \end{aligned} \tag{2.9}$$

In this inequality $C^\top A$ is assumed to be a symmetric matrix with minimum eigenvalue $\gamma > 0$. For example, if $C^\top = I$ is the identity matrix, and A is symmetric with minimum eigenvalue $\gamma > 0$, then inequality (2.9) is satisfied.

We will now verify that inequality (1.5) holds when $\sigma = 2$. In this case the matrix $C^\top A$ is degenerate, and so we will consider a decomposition of the space $\mathbb{R}^n$ into a right sum: $\mathbb{R}^n = H_1 + H_2$, where H_1 is the kernel of the matrix $C^\top A$, and H_2 is the orthogonal complement to H_1. In that case any vector $w - v^* \in \mathbb{R}^n$ has the representation $w - v^* = h_1 + h_2$, where $h_1 = \pi_{H_1}(w - v^*)$ and $h_2 = \pi_{H_2}(w - v^*)$; also, $C^\top A h_1 = 0$, $C^\top A h_2 \in H_2$. On this basis, we will repeat the inequality (2.9) in the degenerate case:

$$\begin{aligned} &[\Phi(w,w) - \Phi(\pi_{\Omega^*}(w), w)] - [\Phi(w, \pi_{\Omega^*}(w)) - \Phi(\pi_{\Omega^*}(w), \pi_{\Omega^*}(w))] = \\ = \;& \langle C^\top A(w-v^*), w-v^*\rangle = \langle (C^\top A)^{1/2}(w-v^*), (C^\top A)^{1/2}(w-v^*)\rangle = \\ = \;& \langle (C^\top A)^{1/2}(h_1+h_2), (C^\top A)^{1/2}(h_1+h_2)\rangle = \langle (C^\top A)^{1/2}h_2, (C^\top A)^{1/2}h_2\rangle = \\ = \;& \langle C^\top A h_2, h_2\rangle \ge \mu|h_2|^2 = \mu|w - v^* - h_1|^2 = \mu|w - v^* - \pi_{H_1}(w-v^*)|^2 = \\ = \;& \mu|w - v^* - \pi_{H_1}(w) + \pi_{H_1}(v^*)|^2 = \mu|w - \pi_{H_1}(w)|^2. \end{aligned}$$

Here μ is the minimum non-zero eigenvalue of the degenerate matrix $C^\top A$. In this chain of argument we have also made use of the fact that the symmetric matrix $(C^\top A)^{1/2}$ has a square root and the projection operators $\pi_{H_1}(w - v^*)$ are linear, where $\pi_{H_1}(v^*) = v^*$.

3. PROXIMAL METHODS FOR CALCULATING THE FIXED POINT OF AN EXTREMAL MAPPING

We will now discuss the proximal control method for solving the extremal inclusion

$$v^* \in \operatorname{Argmin}\{\Phi(w,v^*) : w \in \Omega\}, \qquad v \in \Omega.$$

If v^* is the point of a minimum of $\Phi(w, v^*)$ on the set $w \in \Omega$, it is easy to see that v^* remains a fixed point of the proximal mapping [11]:

$$v^* = \text{argmin}\{0.5|w - v^*|^2 + \alpha\Phi(w, v^*) : \; w \in \Omega\}, \tag{3.1}$$

that is, the proximal step from the point v^* again leads to the same point. Since the proximal operator on the right-hand side of system (3.1) is a non-expandable operator, under certain conditions one can expect the sequence v^n, generated by the process

$$v^{n+1} = \text{argmin}\{0.5|w - v^n|^2 + \alpha\Phi(w, v^n) : \; w \in \Omega\}, \tag{3.2}$$

to converge to the solution of the initial problem. This does in fact happen for optimization problems, but not for equilibrium problems. Convergence of the proximal process (3.2) for equilibrium problems can be guaranteed using the idea of feedback control of the process.

In the general case, feedback [12] is a function which depends on the approximations v^n and the distance between them: $v^{n+1} - v^n$, that is, $u = u(v^n, v^{n+1} - v^n)$. The feedback is zero at equilibrium points: $u = u(v^*, 0) = 0$.

We will introduce the additive control u into system (3.2):

$$v^{n+1} = \text{argmin}\{0.5|w - v^n|^2 + \alpha\Phi(w, v^n + u) : \; w \in \Omega\}. \tag{3.3}$$

We will now formulate the problem of the control of this system: in a certain class of the feedback $u = u(v^n, v^{n+1} - v^n)$, it is required to select a control as a function of the state of system (3.3) which will ensure that the process (3.3) converges to the equilibrium v^*. The simplest controls have the form [11, 13]

$$u = v^{n+1} - v^n. \tag{3.4}$$

If system (3.3) is closed by control (3.4):

$$v^{n+1} = \text{argmin}\{0.5|w - v^n|^2 + \alpha\Phi(w, v^{n+1}) : \; w \in \Omega\}, \tag{3.5}$$

we obtain an implicit iterative system (which is not solved for v^{n+1}) [14, 15], causing certain difficulties in its numerical solution. System (3.5) is an inverse optimization problem and has been analysed in [1]. Thus, we will be interested in feedback corresponding to explicit closed iterative processes. As feedback of that kind, we consider control by the discrepancy [11, 13]

$$u = \text{argmin}\{0.5|w - v^n|^2 + \alpha\Phi(w, v^n) : \; w \in \Omega\} - v^n. \tag{3.6}$$

When system (3.3) has been closed by the feedback (3.6), we obtain an explicit iterative process of the form

$$\begin{aligned} \bar{u}^n &= \text{argmin}\{0.5|w - v^n|^2 + \alpha\Phi(w, v^n) : \; w \in \Omega\}, \\ v^{n+1} &= \text{argmin}\{0.5|w - v^n|^2 + \alpha\Phi(w, \bar{u}^n) : \; w \in \Omega\}. \end{aligned} \tag{3.7}$$

This is clearly an explicit iterative scheme with a preliminary (or predictive) step, as a result of which the prediction $\bar{u}^n$ is calculated first, followed then by the next approximation v^{n+1}.

For simplicity it will always be assumed below that the zero approximation $w^0 \in \{w : |w - \pi_{\Omega^*}(w)| \le 1, \; w \in \Omega\}$.

The convergence theorems proved here are based on the inequality of [16], which is satisfied for any convex, not necessarily differentiable function $f(y)$ defined on the set Q:

$$0.5|z_x - x|^2 + \alpha f(z_x) \le 0.5|z - x|^2 + \alpha f(z) - 0.5|z - z_x|^2, \tag{3.8}$$

where $z \in Q$, and z_x is the point of a minimum of $\varphi_x(x) = 0.5|z-x|^2 + \alpha f(z)$ on Q for a fixed vector x. The truth of this inequality can be established by simple arguments: let z_x be the point of a minimum of $\varphi_x(x)$ on Q. Then by the necessary condition, the subdifferential $\partial\varphi_x(z)$, calculated at the point of the minimum z_x, will contain the positive subgradient

$$\langle z_x - x + \alpha\nabla f(z_x), z - z_x\rangle \geq 0 \qquad \forall\, z \in Q. \tag{3.9}$$

We add the convexity inequality for $f(z)$

$$f(z) \geq f(z_x) + \langle\partial\varphi(z_x), z - z_x\rangle$$

to the identity

$$0.5|z-x|^2 = 0.5|z-z_x|^2 + \langle z - z_x, z_x - x\rangle + 0.5|z_x - x|^2.$$

Then the inequality (3.8) is obtained using (3.9).

A necessary element of the proof of convergence of almost any method, apart from inequality (3.8), is a Lipschitz type inequality. For the function of two variables $\Phi(w,v)$, this inequality can be stated in the form [10]

$$|[\Phi(w+h, v+k) - \Phi(w, v+k)] - [\Phi(w+h, v) - \Phi(w,v)]| \leq |\Phi|\,|h|\,|k|. \tag{3.10}$$

The inequality is true for all w and $w+h$, v and $v+k$ of Ω, and $|\Phi|$ is a constant. The symmetric inequality to (3.10),

$$|[\Phi(w+h, v+k) - \Phi(w+h, v)] - [\Phi(w, v+k) - \Phi(w,v)]| \leq |\Phi|\,|h|\,|k|$$

is also true for all w and $w+h$, v and $v+k$ of Ω, and the constant $|\Phi|$ is different from the analogous constant in (3.10). The classes of functions of two variables which satisfy this condition are non-empty. For example, if $\Phi(w,v)$ is a differentiable function whose partial derivative with respect to the variable w satisfies the Lipschitz condition with constant $|\Phi|$, then inequality (3.10) holds for all w and $w+h$ and v and $v+k$ of Ω. This can be understood easily from the following argument. We use Lagrange's formula

$$f(x+h) - f(x) = \int_0^1 \langle\nabla f(x+th), h\rangle dt$$

and perform the obvious transformations:

$$\begin{aligned} & |[\Phi(w+h, v+k) - \Phi(x, v+k)] - [\Phi(x+h, w) - \Phi(x,w)]| = \\ = & \left|\int_0^1 \langle\nabla\Phi_w(w+th, v+k), h\rangle dt - \int_0^1 \langle\nabla\Phi_w(w+th, v), h\rangle dt\right| \leq \\ \leq & \int_0^1 |\langle\nabla\Phi_w(w+th, v+k) - \nabla\Phi_w(w+th, v), h\rangle| dt \leq \\ \leq & \int_0^1 |\Phi|\,|k|\,|h| dt \leq |\Phi|\,|h|\,|k|. \end{aligned}$$

This proves the assertion.

Since the objective function on the right-hand sides of process (3.7) has the structure of $\varphi(y)$, we can rewrite both equations of (3.7) in an equivalent form to (3.8): $\forall\, w \in \Omega$

$$0.5|v^{n+1} - v^n|^2 + \alpha\Phi(v^{n+1}, \bar{u}^n) \le 0.5|w - v^n|^2 + \alpha\Phi(w, \bar{u}^n) - 0.5|v^{n+1} - w|^2, \quad (3.11)$$

$$0.5|\bar{u}^n - v^n|^2 + \alpha\Phi(\bar{u}^n, v^n) \le 0.5|w - v^n|^2 + \alpha\Phi(w, v^n) - 0.5|\bar{u}^n - w|^2. \quad (3.12)$$

We will now find bounds for the quantity $|\bar{u}^n - v^{n+1}|$. We put $w = \bar{u}^n$ in (3.11) and $w = v^{n+1}$ in (3.12) and add the two inequalities; then, from (3.10), we have

$$\begin{aligned} |v^{n+1} - \bar{u}^n|^2 &\le \alpha(\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^{n+1}, \bar{u}^n) + \Phi(v^{n+1}, v^n) - \Phi(\bar{u}^n, v^n)) \le \\ &\le \alpha|\Phi|\,|v^{n+1} - \bar{u}^n|\,|v^n - \bar{u}^n|. \end{aligned}$$

It follows that

$$|v^{n+1} - \bar{u}^n| \le \alpha|\Phi|\,|v^n - \bar{u}^n|, \quad (3.13)$$

where $|\Phi|$ is the constant of (3.10).

Finally, we represent the initial problem (1.1) in the form of the equivalent variational inequality:

$$\Phi(v^*, v^*) \le \Phi(w, v^*) \qquad \forall\, w \in \Omega. \quad (3.14)$$

We now have all the technical means at our disposal and can proceed to investigate the convergence properties of the process (3.7). The convergence of the process (3.5) is established in the same way.

4. CONVERGENCE TO SHARP EQUILIBRIUM

We will investigate the behaviour of the iterative proximal method depending on the type of equilibrium solution of the original problem, considering the cases of sharp, strongly convex and degenerate equilibria in succession.

Suppose that inequality (1.3) is satisfied with the parameter value $\sigma = 1$. Then

$$\Phi(w, w) - \Phi(\pi_{\Omega^*}(w), w) \ge \gamma|w - \pi_{\Omega^*}(w)| \quad (4.1)$$

for all $w \in \{w : |w - \pi_{\Omega^*}(w)| < 1,\ w \in \Omega\}$, $\gamma > 0$. In that case, problem (1.1) has a set of sharp equilibria, and the process (3.7) converges to one of these after a finite number of steps.

Theorem 1. *If the set of solutions of problem* (1.1) *is non-empty and satisfies the sharp condition* (4.1), *the objective function* $\Phi(w, v)$, $w \in \Omega$, $v \in \Omega$, *is convex with respect to the variable* w *for each fixed value of* v, Ω *is a convex closed set and, in addition, the function* $\Phi(w, v)$ *satisfies condition* (3.10), *then the sequence* v^n *of the proximal process* (3.7) *with parameter* $\alpha < (\sqrt{2}|\Phi|)^{-1}$, *where* $|\Phi|$ *is the constant of* (3.10), *converges after a finite number of iterations to one of the equilibria, that is, there is a number* n_f *such that* $\bar{u}^{n_f} = v^* \in \Omega^*$.

Proof. Put $w = v^* \in \Omega^*$ in (3.11) and $w = v^{n+1}$ in (3.12):

$$0.5|v^{n+1} - v^n|^2 + \alpha\Phi(v^{n+1}, \bar{u}^n) \le 0.5|v^* - v^n|^2 + \alpha\Phi(v^*, \bar{u}^n) - 0.5|v^{n+1} - v^*|^2,$$

$$0.5|\bar{u}^n - v^n|^2 + \alpha\Phi(\bar{u}^n, v^n) \le 0.5|v^{n+1} - v^n|^2 + \alpha\Phi(v^{n+1}, v^n) - 0.5|v^{n+1} - \bar{u}^n|^2.$$

Adding the two inequalities, we obtain

$$\begin{aligned} &|v^{n+1} - v^*|^2 + |v^{n+1} - \bar{u}^n|^2 + |\bar{u}^n - v^n|^2 + \\ +\ &2\alpha\{\Phi(v^{n+1}, \bar{u}^n) - \Phi(\bar{u}^n, \bar{u}^n) + \Phi(\bar{u}^n, v^n) - \Phi(v^{n+1}, v^n)\} + \\ +\ &2\alpha\{\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n)\} \le |v^n - v^*|^2 \qquad \forall\, v^* \in \Omega^*. \end{aligned}$$

Using (3.10) and (3.13), we transform this inequality to obtain:

$$\begin{aligned} & |v^{n+1} - v^*|^2 + |v^{n+1} - \bar{u}^n|^2 + (1 - 2\alpha^2|\Phi|^2)|\bar{u}^n - v^n) + \\ + \quad & 2\alpha\{\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n)\} \leq |v^n - v^*|^2. \end{aligned} \tag{4.2}$$

Since $\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n) \geq 0$, we have

$$|v^{n+1} - v^*|^2 + |v^{n+1} - \bar{u}^n|^2 + d|\bar{u}^n - v^n|^2 \leq |v^n - v^*|^2. \tag{4.3}$$

Here $d = 1 - 2\alpha^2|\Phi|^2 > 0$, since $\alpha < (\sqrt{2}|\Phi|)^{-1}$. Using the inequality

$$0.5|v^{n+1} - v^n|^2 \leq |v^{n+1} - \bar{u}^n|^2 + |\bar{u}^n - v^n|^2, \tag{4.4}$$

from (4.3) we obtain the inequality

$$|v^{n+1} - v^*|^2 + 0.5d|v^{n+1} - v^n|^2 \leq |v^n - v^*|^2. \tag{4.5}$$

We now sum the inequality from $n = 0$ to $n = N$:

$$|v^{N+1} - v^*|^2 + 0.5d \sum_{k=0}^{k=N} |v^{k+1} - v^k|^2 \leq |v^0 - v^*|^2.$$

The partial sums on the left-hand side of this inequality are bounded for any N. As a result, the series $\sum_{k=0}^{\infty} |v^{k+1} - v^k|^2 < \infty$. It follows that

$$|v^{n+1} - v^n| \to 0 \quad \text{as} \quad n \to \infty. \tag{4.6}$$

We now sum (4.3) from $n = 0$ to $n = N$:

$$|v^{N+1} - v^*|^2 + \sum_{k=0}^{k=N} |v^{k+1} - \bar{u}^k|^2 + d \sum_{k=0}^{k=N} |\bar{u}^k - v^k|^2 \leq |v^0 - v^*|^2.$$

It follows from this that the series converge and, therefore, that:

$$|v^{n+1} - \bar{u}^n|^2 \to 0, \quad |\bar{u}^n - v^n|^2 \to 0, \qquad \text{as} \quad n \to \infty. \tag{4.7}$$

We now represent the difference of squares in (4.5) in the form of a product:

$$(|v^{n+1} - v^*| - |v^n - v^*|)(|v^{n+1} - v^*| + |v^n - v^*|) + 0.5d|v^{n+1} - v^n|^2 \leq 0.$$

Thus, allowing for the monotonicity, that is, $|v^{n+1} - v^*| \leq |v^n - v^*|$, we obtain the relation

$$|v^{n+1} - v^*| - |v^n - v^*| + \frac{1}{4}d\frac{|v^{n+1} - v^n|^2}{|v^n - v^*|} \leq 0.$$

Since the relation is true for all $v^* \in \Omega^*$, we put $v^* = \pi_{\Omega^*}(v^n)$ and, since $|v^{n+1} - \pi_{\Omega^*}(v^{n+1})| \leq$ $\leq |v^{n+1} - \pi_{\Omega^*}(v^n)|$, we rewrite this inequality in the form

$$|v^{n+1} - \pi_{\Omega^*}(v^{n+1})| + \frac{1}{4}d\frac{|v^{n+1} - v^n|^2}{|v^n - \pi_{\Omega^*}(v^n)|} \leq |v^n - \pi_{\Omega^*}(v^n)|.$$

We now sum it from $n = 0$ to $n = N$:

$$|v^{N+1} - \pi_{\Omega^*}(v^{N+1})| + \frac{1}{4}d \sum_{k=0}^{k=N} \frac{|v^{k+1} - v^k|^2}{|v^k - \pi_{\Omega^*}(v^0)|} \leq |v^n - \pi_{\Omega^*}(v^n)|.$$

It follows that the series

$$\sum_{k=0}^{\infty} \frac{|v^{k+1} - v^k|^2}{|v^k - \pi_{\Omega^*}(v^k)|} < \infty.$$

Thus

$$\frac{|v^{n+1} - v^n|^2}{|v^n - \pi_{\Omega^*}(v^n)|} \to 0 \quad \text{as} \quad n \to \infty.$$

Using the triangle inequality, we obtain

$$\frac{|v^{n+1} - v^n|^2}{|v^n - \bar{u}^n| + |\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)| + |\pi_{\Omega^*}(\bar{u}^n) - \pi_{\Omega^*}(v^n)|} \leq \frac{|v^{n+1} - v^n|^2}{|v^n - \pi_{\Omega^*}(v^n)|} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence, allowing for (4.7), we have

$$\frac{|v^{n+1} - v^n|^2}{|\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)|} \to 0 \quad \text{as} \quad n \to \infty. \tag{4.8}$$

We now turn to inequality (4.2). Applying the identity

$$|v^{n+1} - v^*|^2 = |v^{n+1} - v^n|^2 + 2\langle v^{n+1} - v^n, v^n - v^* \rangle + |v^n - v^*|^2,$$

we transform the difference of squares:

$$2\langle v^{n+1} - v^n, v^n - v^* \rangle + |v^{n+1} - v^n|^2 + |v^{n+1} - \bar{u}^n|^2 + d|\bar{u}^n - v^n|^2 + 2\alpha[\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n)] \leq 0.$$

Hence

$$\begin{aligned} & 2\langle v^{n+1} - v^n, \bar{u}^n - v^* \rangle + 2\alpha[\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n)] + \\ + \quad & 2\langle v^{n+1} - v^n, v^n - \bar{u}^n \rangle + d|\bar{u}^n - v^n|^2 + |v^{n+1} - v^n|^2 + |v^{n+1} - \bar{u}^n|^2 \leq 0. \end{aligned}$$

We separate the complete square from the third and fourth terms:

$$\begin{aligned} & 2\langle v^{n+1} - v^n, \bar{u}^n - v^* \rangle + 2\alpha[\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n)] + \\ + \quad & |d^{-1/2}(v^{n+1} - v^n) + \sqrt{d}(v^n - \bar{u}^n)|^2 + \\ + \quad & (1 - 1/d)||v^{n+1} - v^n|^2 + |v^{n+1} - \bar{u}^n|^2 \leq 0. \end{aligned}$$

It follows from the last inequality that

$$2\langle v^{n+1} - v^n, \bar{u}^n - v^* \rangle + 2\alpha[\Phi(\bar{u}^n, \bar{u}^n) - \Phi(v^*, \bar{u}^n)] \leq (1/d - 1)||v^{n+1} - v^n|^2.$$

Since this inequality is true for all $v^* \in \Omega^*$, if $v^* = \pi_{\Omega^*}(\bar{u}^n)$ and the sharp condition (4.1) holds, we obtain

$$2\alpha\gamma|\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)| \leq 2|v^{n+1} - v^n|\,|\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)| + (1/d - 1)|v^{n+1} - v^n|^2.$$

Assuming that $|\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)| \neq 0$ for all n, we obtain

$$\alpha\gamma \leq |v^{n+1} - v^n| + (1/d - 1)\frac{v^{n+1} - v^n|^2}{|\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)|}. \tag{4.9}$$

It can be seen by comparing inequalities (4.5), (4.8) and (4.9) that they contradict one another. Hence, the assumption that $|\bar{u}^n - \pi_{\Omega^*}(\bar{u}^n)| \neq 0$ for all n is false, and there is, therefore, a number n_f such that $\bar{u}^{n_f} = \pi_{\Omega^*}(\bar{u}^{n_f}) = v^* \in \Omega^*$. This proves the theorem. □

5. CONVERGENCE TO STRONGLY CONVEX EQUILIBRIUM

In this section we obtain estimates for the rate of convergence to strongly convex equilibrium of the process (3.7).

Let the solution of the initial problem be unique, and let the behaviour of the objective function in its neighborhood be described by an inequality of the form

$$[\Phi(w,w)-\Phi(v^*,w)]-[\Phi(w,v^*)-\Phi(v^*,v^*)]\geq\gamma|w-v^*|^2 \tag{5.1}$$

for all $w\in\{w:|w-v^*|<1,\ w\in\mathbb{R}^n\}$, where the parameter $\gamma>0$.

Theorem 2. *If the set of solutions of problem* (1.1) *is non-empty and satisfies condition* (5.1)*, the objective function* $\Phi(w,v)$*,* $w\in\Omega$*,* $v\in\Omega$*, is convex with respect to the variable* w *for each fixed value of* v*,* $\Omega\in\mathbb{R}^n$ *is a convex closed set and, in addition, the function* $\Phi(w,v)$ *satisfies condition* (3.10)*, then the sequence* v^n *of the proximal process* (3.7) *with parameter* $\alpha<$ $<(\sqrt{2}|\Phi|)^{-1}$*, where* $|\Phi|$ *is the constant in* (3.10)*, converges at the rate of a geometric progression to one of the equilibria, that is,*

$$|v^{n+1}-v^*|^2\leq q(\alpha)^{n+1}|v^0-v^*|^2 \quad as \quad n\to\infty,$$

where $q(\alpha)=[1+4(\alpha\gamma)^2/d-2\alpha\gamma]<1$*,* $d=1+2\alpha\gamma-2\alpha^2|\Phi|^2$*.*

Proof. We put $w=v^*\in\Omega^*$ in (3.11), $w=v^{n+1}$ in (3.12) and $w=\bar{u}^n$ in (3.14). Then

$$\begin{aligned}
&0.5|v^{n+1}-v^n|^2+\alpha\Phi(v^{n+1},\bar{u}^n)\leq 0.5|v^*-v^n|^2+\alpha\Phi(v^*,\bar{u}^n)-0.5|v^{n+1}-v^*|^2,\\
&0.5|\bar{u}^n-v^n|^2+\alpha\Phi(\bar{u}^n,v^n)\leq 0.5|v^{n+1}-v^n|^2+\alpha\Phi(v^{n+1},v^n)-0.5|v^{n+1}-\bar{u}^n|^2,\\
&\Phi(v^*,v^*)\leq\Phi(\bar{u}^n,v^*).
\end{aligned}$$

We add all three inequalities. Then

$$\begin{aligned}
&|v^{n+1}-v^*|^2+|v^{n+1}-\bar{u}^n|^2+|\bar{u}^n-v^n|^2)+\\
+\ &2\alpha[\Phi(v^{n+1},\bar{u}^n)-\Phi(\bar{u}^n,\bar{u}^n)+\Phi(\bar{u}^n,v^n)-\Phi(v^{n+1},v^n)]+\\
+\ &2\alpha\{[\Phi(\bar{u}^n,\bar{u}^n)-\Phi(v^*,\bar{u}^n)]-[\Phi(\bar{u}^n,v^*)-\Phi(v^*,v^*)]\leq\\
\leq\ &|v^n-v^*|^2 \qquad \forall\, v^*\in\Omega^*.
\end{aligned}$$

Taking (3.10) and (3.13) into account, we obtain

$$\begin{aligned}
&|v^{n+1}-v^*|^2+|v^{n+1}-\bar{u}^n|^2+(1-2\alpha^2|\Phi|^2)|\bar{u}^n-v^n|^2+\\
+\ &2\alpha\{[\Phi(\bar{u}^n,\bar{u}^n)-\Phi(v^*,\bar{u}^n)]-[\Phi(\bar{u}^n,v^*)-\Phi(v^*,v^*)]\}\leq|v^n-v^*|^2.
\end{aligned} \tag{5.2}$$

Allowing for the sharp minimum property of (5.1), we have

$$|v^{n+1}-v^*|^2+|v^{n+1}-\bar{u}^n|^2+(1-2\alpha^2|\Phi|^2)|\bar{u}^n-v^n|^2+2\alpha\gamma|\bar{u}^n-v^*|\leq|v^n-v^*|^2. \tag{5.3}$$

Using the identity

$$|\bar{u}^n-v^*|^2=|\bar{u}^n-v^n|^2+2\langle\bar{u}^n-v^n,v^n-v^*\rangle+|v^n-v^*|^2$$

we transform (5.2):

$$\begin{aligned}
|v^{n+1}-v^*|^2\ &+\ |v^{n+1}-\bar{u}^n|^2+(1-2\alpha\gamma-2\alpha^2|\Phi|^2)|\bar{u}^n-v^n|^2+\\
&+\ 4\alpha\gamma\langle\bar{u}^n-v^n,v^n-v^*\rangle+2\alpha\gamma|v^n-v^*|^2\leq|v^n-v^*|^2.
\end{aligned}$$

We separate the complete square from the third and fourth terms:

$$\begin{aligned} & |v^{n+1} - v^*|^2 + |v^{n+1} - \bar{u}^n|^2 + \\ + \quad & \left|\sqrt{d}(\bar{u}^n - v^n) + \frac{2\alpha\gamma}{\sqrt{d}}(v^n - v^*)\right|^2 - \frac{(2\alpha\gamma)^2}{d}|v^n - v^*|^2 + \\ + \quad & 2\alpha\gamma|v^n - v^*|^2 \leq |v^n - v^*|^2, \end{aligned}$$

where $d = 1 + 2\alpha\gamma - 2\alpha^2|\Phi|^2$. Hence

$$|v^{n+1} - v^*|^2 \leq (1 + 4(\alpha\gamma)^2/d - 2\alpha\gamma)|v^n - v^*|^2.$$

Since $\alpha < (\sqrt{2}|\Phi|^2)^{-1}$, we have $q(\alpha) = (1 + 4(\alpha\gamma)^2/d - 2\alpha\gamma) < 1$.

Thus,

$$|v^{n+1} - v^*|^2 \leq q(\alpha)|v^n - v^*|^2.$$

Summing this inequality, we obtain

$$|v^{n+1} - v^*|^2 \leq q(\alpha)^{n+1}|v^0 - v^*|^2,$$

where the parameter satisfies the condition $\alpha < (\sqrt{2}|\Phi|)^{-1}$. This proves the theorem. □

6. CONVERGENCE TO DEGENERATE EQUILIBRIUM

We will now consider the convergence of the proximal process (3.7) to degenerate equilibrium, which is described by an inequality of the form

$$[\Phi(w, w) - \Phi(v^*, w)] - [\Phi(w, v^*) - \Phi(v^*, v^*)] \geq 0 \tag{6.1}$$

for all $w \in S = \{w : w - \pi_{\Omega^*}(w)| < 1,\ w \in \mathbb{R}^n\}$, where Ω^* is the convex closed set of solutions of problem (1.1). We will show in that case that the process (3.7) converges monotonely with respect to the norm to one of the equilibrium solutions.

Theorem 3. *If the set of solutions of problem* (1.1) *is non-empty and satisfies condition* (6.1), *the objective function* $\Phi(w, v)$, $w \in \Omega$, $v \in \Omega$, *is convex with respect to the variable* w *for each fixed value of* v, $\Omega \in \mathbb{R}^n$ *is a convex closed set and, in addition, the function* $\Phi(w, v)$ *satisfies condition* (3.10), *then the sequence* v^n *of the proximal process* (3.7) *with parameter* $\alpha < (\sqrt{2}|\Phi|)^{-1}$, *where* $|\Phi|$ *is the constant of* (3.10), *converges monotonely with respect to the norm to one of the equilibrium solutions, that is,* $v^n \to v^* \in \Omega^*$ *as* $n \to \infty$ *for all* $v^0 \in \mathbb{R}^n$.

Proof. From inequality (5.2), allowing for condition (6.1), we obtain

$$|v^{n+1} - v^*|^2 + |v^{n+1} - \bar{u}^n|^2 + (1 - 2\alpha^2|\Phi|^2)|\bar{u}^n - v^n|^2 \leq |v^n - v^*|^2. \tag{6.2}$$

Assuming that the parameter α is chosen from the condition $\alpha < (\sqrt{2}|\Phi|)^{-1}$ (in which case $d = (1 - 2\alpha^2|\Phi|^2) > 0$, we sum the inequality (6.2) from $n = 0$ to $n = N$:

$$|v^{N+1} - v^*|^2 + d\sum_{k=0}^{k=N}|v^{k+1} - \bar{u}^k|^2 + d\sum_{k=0}^{k=N}|\bar{u}^k - v^k|^2 \leq |v^0 - v^*|^2.$$

It follows from this inequality that the trajectory is bounded:

$$|v^{N+1} - v^*|^2 \leq |v^0 - v^*|^2,$$

and also that the series converge:

$$\sum_{k=0}^{\infty} |v^{k+1} - \bar{u}^k|^2 < \infty, \qquad \sum_{k=0}^{\infty} |\bar{u}^k - v^k|^2 < \infty$$

and, therefore, that

$$|v^{n+1} - \bar{u}^n|^2 \to 0, \qquad |\bar{u}^n - v^n|^2 \to 0 \quad \text{as} \quad n \to \infty$$

Thus, using (4.4), we obtain $|v^{n+1} - v^n| \to 0$ as $n \to \infty$.

Since the sequence v^n is bounded, there is an element v' such that $v^{n_i} \to v'$ as $n_i \to \infty$, and

$$|v^{n_i+1} - \bar{u}^{n_i}|^2 \to 0, \qquad |\bar{u}^{n_i} - v^{n_i}|^2 \to 0.$$

We now consider inequality (3.11) for all $n_i \to \infty$ and, taking the limit, write out the limiting inequality

$$\Phi(v', v') \leq \Phi(w, v') \qquad \forall\, w \in \Omega.$$

This inequality is obviously the same as (3.14) and, therefore, $v' = v^* \in \Omega^*$. Thus, any limit point of the sequence v^n is a solution of the problem, and the quantity $|v^n - v^*|^2$ decreases monotonely. These two facts together mean that the sequence v^n can have only one limit point, that is, v^n converges monotonely with respect to the norm to one of the solutions of the problem: $v^n \to v^*$ as $n \to \infty$. This proves the theorem. □

All three theorems proved above also apply to the implicit process (3.5).

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