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Extragradient approach to solution of two person non-zero sum games

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Abstract. The two person non-zero sum game is considered in the following statements: in classical form, under the availability of inequality-constraints and coupled constraints. A extragradient method for computing Nash equilibria for these statements is suggested and its convergence is investigated.

Keywords: Non-zero sum game, extragradient method, coupled constraints, convergence.

1 Introduction

Let $\Omega = X_1 \times X_2$ be a rectangle, where $X_1 \in R_1^{n_1}$, $X_2 \in R_2^{n_2}$ are convex closed sets from finite-dimensional Euclidean spaces, generally speaking, various dimensionality. Let functions $f_1(x_1, x_2) + \varphi_1(x_1)$, $f_2(x_1, x_2) + \varphi_2(x_2)$ be determined on the product space $R_1^{n_1} \times R_2^{n_2}$. Consider a extreme mapping $Y(x) = y_1(x_2) \times y_2(x_1)$, which maps any point $x = (x_1, x_2) \in \Omega$ to some convex closed subset from Ω

$$\begin{aligned} y_1(x_2) &\in \text{Argmin}\{f_1(z_1, x_2) + \varphi_1(z_1) \mid z_1 \in X_1\}, \\ y_2(x_1) &\in \text{Argmin}\{f_2(x_1, z_2) + \varphi_2(z_2) \mid z_2 \in X_2\}. \end{aligned} \tag{1}$$

The subset $Y(x)$ represents by itself the direct product of optimal solution sets for problem (1). If functions $f_1(z_1, x_2) + \varphi(z_1)$, $f_2(x_1, z_2) + \varphi(z_2)$ are continuous and convex in own variables, i.e. the first function is convex in z_1, the second one is convex in z_2 for any x_1 and x_2, X_i, $i = 1, 2$ are convex compact sets, then there exists a fixed point $x^* = (x_1^*, x_2^*)$ of this mapping, Aubin, Frankowska 1990. This point satisfies the system of extreme inclusions

$$\begin{aligned} x_1^* &\in \text{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) \mid z_1 \in X_1\}, \\ x_2^* &\in \text{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) \mid z_2 \in X_2\}. \end{aligned} \tag{2}$$

The problem of computing a fixed point (2) was formulated by Nash 1950 as a n-person game. The (2) is its particular case. In this game the set X_i, $i = 1, 2$ is interpreted as the strategy set of i-th player, where $z_i \in X_i$ is a individual strategy of the player, $f_1(z_1, x_2) + \varphi_1(z_1)$ and $f_2(x_1, z_2) + \varphi_2(z_2)$ are payoff functions of players. A characteristic sense of a fixed point x^* of (2) is that any player is not interested to disturb the equilibrium state as anybody of them cannot reduce value of its payoff function in the one-sided order. Any solution of game (2) we shall call also as equilibrium solution or equilibrium. This solution, in particular, means a condition of compromise with a summarized prize $f_1(x_1^*, x_2^*) + \varphi_1(x_1^*) + f_2(x_1^*, x_2^*) + \varphi_2(x_2^*)$.

After publication, Nash 1950 the efforts of researchers were undertaken to develop solution methods for games. As the result the methods of solving for two person zero-sum game were found out. Formally this game can be reduced to a saddle point problem and, therefore, saddle point methods provide some tools to solve a zero-sum game. We mark some papers in this topic. First of all it is the iterative method for an evaluation of value of matrix game. The method was offered by Brown 1951, the convergence of it was proved by Robinson 1951. Bakushinsky and Goncharsky 1994 extended the approach to convex-concave games and proved convergence of it for two person zero-sum games. The number of the approaches uses idea of an transformation of game to other type of problems. For example, Hansen and Scarf 1974 approximated a game by means of some mapping to calculate a fixed point of it with the help of simplicial partitions of a set. Lemke and Hawson 1964 reduced the game to complementarity problem and applied the pivoting-approach to solve it. However from a point of view of game mathematical modelling the game methods look more attractive on a comparison with approaches of a reduction them to other types of problems. We pick out another paper, Mills 1960, where a bimatrix game is reduced to non-convex quadratic programming problem.

In the present article the gradient descent idea for the solution of non-zero sum game is used. Earlier, this gradient idea but in other form was considered and justified for potential game by Rosen 1965. The outcomes obtained there follow as a particular case from outcomes of the present paper. It is known, Nikaido and Isoda 1955 that any game always can be presented in equivalent form of computing a fixed point of extreme inclusion induced by normalized function of two vectorial variables of the same dimensionality. If the initial problem was a zero-sum game, then the normalized function is antisymmetric. In this paper the sizeable extension of class of games with an antisymmetric normalized function (i.e. zero-sum games) is introduced up to a game class with positive semidefinite (skew-symmetric) normalized function. The introduced class of game problems includes itself all zero-sum games and this class can be considered as analog of convex programming in the class of non-linear programming problems. The theory of solution methods for bilinear games is developed and presented in Antipin 2002.

2 Discussion of the problem

Problem (2) represents a system of extreme inclusions, which it always is possible to scalarize and to present in the form of problem for computing a fixed point of extreme map. To that end we enter a normalized function of the kind

$$\Phi(v, w) + \varphi(w) = f_1(z_1, x_2) + \varphi_1(z_1) + f_2(x_1, z_2) + \varphi_2(z_2),$$

where $w = (z_1, z_2)$, $v = (x_1, x_2)$, $v, w \in \Omega = X_1 \times X_2$. In terms of new variables the problem (2) can be written in the shape

$$v^* \in \text{Argmin}\{\Phi(v^*, w) + \varphi(w) \mid w \in \Omega\} \tag{3}$$

or that is the same

$$\Phi(v^*, v^*) + \varphi(v^*) \leq \Phi(v^*, w) + \varphi(w) \quad \forall w \in \Omega. \tag{4}$$

Uneasy to be convinced of equivalence of problems (2) and (3). Really, we present (4) as

$$\begin{array}{rl} & f_1(x_1^*, x_2^*) + \varphi_1(x_1^*) + f_2(x_1^*, x_2^*) + \varphi_2(x_2^*) \leq \\ \leq & f_1(z_1, x_2^*) + \varphi_1(z_1) + f_2(x_1^*, z_2) + \varphi_2(z_2), \qquad z_1 \in X_1, \ \ z_2 \in X_2. \end{array}$$

By virtue of separability of function $\Phi(v, w)$ and modularity of set Ω the last inequality can be splitted on a system of inequalities

$$\begin{array}{ll} f_1(x_1^*, x_2^*) + \varphi_1(x_1^*) \leq f_2(z_1, x_2^*) + \varphi_1(z_1) & \forall z_1 \in X_1, \\ f_2(x_1^*, x_2^*) + \varphi_2(x_2^*) \leq f_2(x_1^*, z_2) + \varphi_2(z_2) & \forall z_2 \in X_2, \end{array}$$

i.e any fixed point (4) is the solution (2). The inverse proposition is true too.

If game (2) satisfies the condition $f_1(x_1, x_2) + \varphi_1(x_1) + f_2(x_1, x_2) + \varphi_2(x_2) = 0$ $\forall x_1 \in X_1$, $x_2 \in X_2$, then it is called a zero sum game. It follows from this condition immediately that $f_1(x_1, x_2) = -f_2(x_1, x_2) = f(x_1, x_2)$, $\varphi_1(x_1) = 0$, $\varphi_2(x_1) = 0$ $\forall x_1 \in X_1$, $x_2 \in X_2$, i.e.

$$\begin{array}{l} x_1^* \in \text{Argmin}\{\ \ f(z_1, x_2^*) \mid z_1 \in X_1\}, \\ x_2^* \in \text{Argmin}\{-f(x_1^*, z_2) \mid z_2 \in X_2\}. \end{array} \tag{5}$$

Obviously that the problem can be rewritten in the form of a system of inequalities

$$f(x_1^*, z_2) \leq f(x_1^*, x_2^*) \leq f(z_1, x_2^*) \qquad \forall z_1 \in X_1, \ z_2 \in X_2. \tag{6}$$

In this case, pair of vectors x_1^*, x_2^* is the saddle point of $f(x_1, x_2)$ on set $X_1 \times X_2$.

It is useful to mark that problem (3) also can be interpreted as two person game, where the strategies of the first player are described by means of variable $v \in \Omega$, and second one are in variable $w \in \Omega$. The choice of strategy of the first player consists in presentation of specific vector $v \in \Omega$, the response of the second player is in presentation of set $Y(v) \subset \Omega$. In this situation it is required to choice $v = v^*$ such that $v^* \in Y(v^*)$.

We have seen if $v^* = (x_1^*, x_2^*)$ is a fixed point in (3), then the pair of vectors (x_1^*, x_2^*) is the saddle point of function $f(x_1, x_2)$ in (5).

Antipin 2001B exhibits that the saddle point property is the key property to prove the convergence of various methods to fixed points of extreme maps. Therefore the question arises on whether can pair v^*, v^* be a saddle point some function connected with this problem, if v^* is a solution of (3) ? To answer this question we enter a condition, which is enough to describe the situation as a whole.

A function $\Phi(v, w)$ is called positive semi-definite or skew-symmetric on $\Omega \times \Omega$ if it obeys the inequality, Antipin 1995

$$\Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \geq 0 \qquad \forall v, w \in \Omega \times \Omega. \tag{7}$$

This condition can be considered, on the one hand, as generalization of antisymmetric property, i.e. when the condition $\Phi(v, w) = -\Phi(w, v)$ is held, and on the other hand, as generalization of concept of positive semi-definiteness of matrices. Indeed, if a function has a bilinear structure $\Phi(v, w) = \langle \Phi v, w \rangle$, where Φ is a square matrix, then condition (7) takes the form of positive semi-definiteness for matrix, i.e. $\langle \Phi(v - w), v - w \rangle \geq$ $\geq 0 \ \forall (v - w) \in R^n$. Easy to check up that if function $\Phi(v, w)$ is positive semi-definite, then $\Phi(v, w) + \varphi(w)$ is positive semi-definite as well. Positive semi-definite condition (7) allows to mark out the class of equilibrium problems which may be considered as analog of classes of convex and saddle point programming problems.

The equilibrium problems subjected to the condition of positive semi-definiteness have the important properties, namely, if v^* is a solution of equilibrium problem (3), then the pair v^*, v^* is the saddle point of shift function $\Psi(v, w) = \Phi(v, w) + \varphi(w) -$ $-\Phi(v, v) - \varphi(v)$ (it is identically equal to zero on the diagonal of square $\Omega \times \Omega$). Indeed, if v^* is a solution, then at $v = v^*$ from (4) and (7) we have

$$\Phi(w, w) + \varphi(w) - \Phi(w, v^*) - \varphi(v^*) \geq \Phi(v^*, w) + \varphi(w) - \Phi(v^*, v^*) - \varphi(v^*) \geq 0$$

for all $w \in \Omega$. Thence

$$\Psi(v, v^*) \leq \Psi(v^*, v^*) \leq \Psi(v^*, w) \qquad \forall v, w \in \Omega \times \Omega. \tag{8}$$

Thus, the v^*, v^* is the saddle point for the shift function $\Psi(v, w)$. This function is convex in w and, generally speaking, is not concave in v. Below we shall show that saddle point condition of function $\Psi(v, w)$ is crucial property in the substantiation of convergence for many iterative methods to a equilibrium solution of (3).

The v^*, v^* may be a saddle point for another functions, too, for example, for $\Psi_1(v, w) = \langle \nabla_2 \Phi(v, v), w - v \rangle$, where $\nabla_2 \Phi(v, w)$ is partial gradient $\Phi(v, w)$ in w for any v. The last function is very convenient to use it for the substantiation of convergence for gradient-type methods. Therefore we prove this property.

In this paper the function $\Phi(v,w)+\varphi(w)$ is supposed to be convex in w for any v, i.e. it subjects to the system of inequalities

$$\langle \nabla f(x), y-x\rangle \le f(y)-f(x) \le \langle \nabla f(y), y-x\rangle \tag{9}$$

for all x и y from a set. We use this system of inequalities for (7) for the case $\Phi(v,w)+$ $+\varphi(w)$

$$\Phi(w,w)+\varphi(w)-\Phi(w,v)-\varphi(v)-\Phi(v,w)-\varphi(w)+\Phi(v,v)+\varphi(v)\ge 0 \tag{10}$$

for all $v,w\in\Omega\times\Omega$, then we get

$$\langle \nabla_2\Phi(w,w)+\nabla\varphi(w)-\nabla_2\Phi(v,v)+\nabla\varphi(v), w-v\rangle \ge 0 \qquad \forall w,v\in\Omega\times\Omega, \tag{11}$$

i.e. gradient-restriction $(\nabla_2\Phi(v,w)+\nabla\varphi(w))|_{v=w}$ is the monotone operator on the diagonal of square $\Omega\times\Omega$. We shall write out a necessary condition for problem (3) in the form of variational inequality

$$\langle \nabla_2\Phi(v^*,v^*)+\nabla\varphi(v^*), w-v^*\rangle \ge 0 \qquad \forall w\in\Omega. \tag{12}$$

Let us set $v=v^*$ in (11) and compare with (12), then

$$\langle \nabla_2\Phi(w,w)+\nabla\varphi(w), w-v^*\rangle \ge 0 \qquad \forall w\in\Omega. \tag{13}$$

Under notation of function $\Psi_1(v,w)=\langle\nabla_2\Phi(v,v), w-v\rangle$ both inequalities can be recorded in the form of a saddle point condition

$$\Psi_1(v,v^*)\le\Psi_1(v^*,v^*)\le\Psi_1(v^*,w) \qquad \forall v,w\in\Omega\times\Omega.$$

Hereinafter, we shall assume that objective function $\Phi(v,w)+\varphi(w)$ of (3) has two main properties: it is convex in w for any v, and $\Phi(v,w)$ is positive semi-definite. Both properties guarantee the existence of saddle point property for a point v^*,v^* that, in turn, provides convergence of majority iterative processes to equilibrium solutions.

The above reasoning are correct provided that the function $\Phi(v,w)$ is positive semi-definite. In case it is not so, it is true that any function always can be resulted to the positive semi-definite kind. The latter circumstance is very important, as it has universal character. We show that any function $\Phi(v,w)$ can be splitted on two components: symmetric and antisymmetric.

We select two linear subspaces in the linear space of the real-valued functions $\Phi(v,w)$, which are determined by the following conditions

$$\Phi(v,w)-\Phi(w,v) = 0 \qquad \forall w\in\Omega, \quad \forall v\in\Omega, \tag{14}$$
$$\Phi(v,w)+\Phi(w,v) = 0 \qquad \forall w\in\Omega, \quad \forall v\in\Omega. \tag{15}$$

The functions of the first subspace are called symmetric; those of the second class, anti-symmetric.

Recall that a pair of points with coordinates w, v and v, w is situated symmetrically concerning the diagonal of the square $\Omega \times \Omega$, i.e., with respect to the linear manifold $v = w$. This allows us to introduce the concept of a transposed function, Antipin 1998. If we assign the value of $\Phi(w, v)$ calculated at the point w, v to every point with coordinates v, w, that is $v, w \to \Phi(w, v)$, then we obtain the transposed function $\Phi^\top(v, w) = \Phi(w, v)$. In terms of this function conditions (14) and (15) look like

$$\Phi(v, w) = \Phi^\top(v, w), \qquad \Phi(v, w) = -\Phi^\top(v, w).$$

Using the obvious relations $\Phi(v, w) = (\Phi^\top(v, w))^\top$, $(\Phi_1(v, w) + \Phi_2(v, w))^\top = \Phi_1^\top(v, w) + \Phi_2^\top(v, w)$, we can readily verify that any real function $\Phi(v, w)$ can be represented as the sum

$$\Phi(v, w) = S(v, w) + K(v, w), \tag{16}$$

where $S(v, w)$ and $K(v, w)$ are symmetric and antisymmetric functions, respectively. This expansion is unique

$$S(v, w) = \frac{1}{2}\left(\Phi(v, w) + \Phi^\top(v, w)\right), \qquad K(v, w) = \frac{1}{2}\left(\Phi(v, w) - \Phi^\top(v, w)\right). \tag{17}$$

Antipin 1998 had shown that the gradient-restriction $\nabla_2 S(v, w)|_{v=w}$ coincides with the gradient $(1/2)\nabla S(v, v)$ of the restriction for symmetric function $S(v, v) = S(v, w)|_{v=w}$. The latter means that alongside with the function $\Phi(v, w) + \varphi(w)$ it always is possible to introduce other function $K(v, w) + S(w, w) + \varphi(w) = K(v, w) + \varphi_1(w)$ such that gradient-restrictions of both functions are the same, and $K(v, w)$ is a positive semi-definite function. Thus, if it is assumed that $\Phi(v, w) + \varphi(w)$ is not positive semi-definite function, then there always exists an other positive semi-definite function $K(v, w) + \varphi_1(w)$ such that the gradient-restrictions for the both functions are the same and, certainly, a solution sets of both equilibrium problem generated both functions are the same as well. Therefore, it is possible approve that the positive semi-definiteness condition is not restraining very much, though under transition from one function to other there will be difficulties with fulfilment of the second condition: convexity condition of $K(v, w) + \varphi_1(w)$ in w for any v.

3 Extragradient game methods

In game (2) each of the participants has to decide a minimization problem with convex functions in own variable under fixed values of parameters, which are simultaneously variables for each contenders. There is the question, when methods developed for solving convex problems can be transferred for solving a system of such problems.

First of all we note that the game (2) has several equivalent formulations. For example, in the form of variational inequalities

$$\begin{aligned} \langle \nabla_1 f_1(x_1^*, x_2^*) &+ \nabla\varphi_1(x_1^*), z_1 - x_1^* \rangle \geq 0 \qquad \forall z_1 \in X_1, \\ \langle \nabla_2 f_2(x_1^*, x_2^*) &+ \nabla\varphi_2(x_2^*), z_2 - x_2^* \rangle \geq 0 \qquad \forall z_2 \in X_2, \end{aligned} \tag{18}$$

or in the kind of operator equations

$$\begin{aligned} x_1^* &= \pi_{X_1}(x_1^* - \alpha(\nabla_1 f_1(x_1^*, x_2^*) + \nabla\varphi_1(x_1^*))), \\ x_2^* &= \pi_{X_2}(x_2^* - \alpha(\nabla_2 f_2(x_1^*, x_2^*) + \nabla\varphi_2(x_2^*))). \end{aligned} \tag{19}$$

where $\nabla_1 f_1(x_1, x_2)$, $\nabla_2 f_2(x_1, x_2)$ are partial derivatives of functions $f_1(x_1, x_2)$ and $f_2(x_1, x_2)$ in first and second variables accordingly, $\nabla\varphi_1(x_1)$, $\nabla\varphi_2(x_2)$ are gradients of appropriate functions, $\pi_{X_i}(\ldots)$, $i = 1, 2$, are projection operators of vectors onto a set X_i, $\alpha > 0$ is a parameter like steplength. Conditions (18), (19) represent necessary, and in the convex case, sufficient conditions of a minimum in (2). Recall that the solution set of positive semi-definite equilibrium (game) problems introduced above is convex closed set, Antipin 2002

For the solution of system (19) the extragradient method is used. It includes two half-steps Antipin 1998

the first half-step

$$\begin{aligned} \bar{x}_1^n = \pi_{X_1}(x_1^n - \alpha(\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n))), \\ \bar{x}_2^n = \pi_{X_2}(x_2^n - \alpha(\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n))), \end{aligned} \tag{20}$$

and second half-step

$$\begin{aligned} x_1^{n+1} = \pi_{X_1}(x_1^n - \alpha(\nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n))), \\ x_2^{n+1} = \pi_{X_2}(x_2^n - \alpha(\nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n))). \end{aligned} \tag{21}$$

A steplength $\alpha > 0$ in this process is determined from some condition, which will be mentioned below. The first half-step is treated as a calculation of prediction point where direction of the future motion can be determined and then the second half-step implements the motion in computed direction.

Note that the ordinary gradient method like type

$$\begin{aligned} x_1^{n+1} = \pi_{X_1}(x_1^n - \alpha(\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n))), \\ x_2^{n+1} = \pi_{X_2}(x_2^n - \alpha(\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n))) \end{aligned}$$

does not converge to a solution of (19). But, in the particular case, if the operator

$$\nabla_2\Phi(v, w)|_{v=w} = (\nabla_1 f_1(x_1, x_2), \nabla_2 f_2(x_1, x_2))$$

is potential (i.e. Jacobian of $\nabla_2\Phi(v, v)$ is the symmetric matrix), then the ordinary gradient method is converging, Rosen 1965. In this case the equilibrium problem is equivalent to optimization problem, Antipin 2001B.

From (20), (21) we have estimates

$$\begin{aligned} |\bar{x}_1^n - x_1^{n+1}| \le \alpha|\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n) - \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_1(\bar{x}_1^n)|, \\ |\bar{x}_2^n - x_2^{n+1}| \le \alpha|\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n) - \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_2(\bar{x}_2^n)|. \end{aligned} \tag{22}$$

We present process (20), (21) in the form of variational inequalities. Equation (20) rewrites accordingly with definition of the projection operator as

$$\begin{aligned} &\langle \bar{x}_1^n - x_1^n + \alpha(\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n)), z_1 - \bar{x}_1^n\rangle \geq 0 \quad \forall z_1 \in X_1, \\ &\langle \bar{x}_2^n - x_2^n + \alpha(\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n)), z_2 - \bar{x}_2^n\rangle \geq 0 \quad \forall z_2 \in X_2. \end{aligned} \tag{23}$$

Equation (21) presents as well as

$$\begin{aligned} &\langle x_1^{n+1} - x_1^n + \alpha(\nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n)), z_1 - x_1^{n+1}\rangle \geq 0 \quad \forall z_1 \in X_1, \\ &\langle x_2^{n+1} - x_2^n + \alpha(\nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n)), z_2 - x_2^{n+1}\rangle \geq 0 \quad \forall z_2 \in X_2. \end{aligned} \tag{24}$$

Under discussing questions of convergence of game method (20), (21) it is important to underline that two players are a whole system, which in due course evolves to equilibrium state and character of this evolution is determined, in main, by system properties, i.e. system properties of the players, as a whole unit. These system properties we formulate in terms of the normalized function.

$$\Phi(v, w) + \varphi(w) = f_1(z_1, x_2) + \varphi_1(z_1) + f_2(x_1, z_2) + \varphi_2(z_2), \tag{25}$$

где $w = (z_1, z_2)$, $v = (x_1, x_2)$, $v, w \in \Omega = X_1 \times X_2$. They include: the positive semi-definite property of $\Phi(v, w)$, the Lipschitz condition of gradient-restriction of this function and convexity property of $\Phi(v, w)$ in w for any v. Certainly, all these properties, in turn, are determined by properties of functions $f_1(z_1, x_2)$, $\varphi_1(z_1)$, $f_2(x_1, z_2)$, $\varphi_2(z_2)$. For example, if these functions are convex in z_1 and z_2 for any values x_1 and x_2, then $\Phi(v, w) + \varphi(w)$ is convex in w for any v.

Theorem 1 *Suppose that a solution set of game* (2) *is non-empty, normalized function of this game* $\Phi(v, w) + \varphi(w)$ *is positive semi-definite and convex in* w *for any* v, *its gradient-restriction* $\nabla_2\Phi(v, w)|_{v=w} + \nabla\varphi(w)$ *satisfies the Lipschitz condition with constant* L, $\Omega \subseteq R^n$ *is convex closed set. Then, the sequence* x_1^n, x_2^n *generated by method* (20), (21) *with steplength* α *chosen from condition* $0 < \alpha < 1/(\sqrt{2}L)$ *converges to a game solution, i.e.* $x_1^n, x_2^n \to x_1^*, x_2^*$ *as* $n \to \infty$ *monotonically in the norm.*

Proof. By putting $z_1 = x_1^*$, $z_2 = x_2^*$ in (24), then

$$\begin{aligned} &\langle x_1^{n+1} - x_1^n + \alpha(\nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n)), x_1^* - x_1^{n+1}\rangle \geq 0, \\ &\langle x_2^{n+1} - x_2^n + \alpha(\nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n)), x_2^* - x_2^{n+1}\rangle \geq 0. \end{aligned} \tag{26}$$

By setting $z_1 = x_1^{n+1}$, $z_2 = x_2^{n+1}$ in (23)

$$\begin{aligned} &\langle \bar{x}_1^n - x_1^n + \alpha(\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n)), x_1^{n+1} - \bar{x}_1^n\rangle \geq 0, \\ &\langle \bar{x}_2^n - x_2^n + \alpha(\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(\bar{x}_2^n)), x_2^{n+1} - \bar{x}_2^n\rangle \geq 0. \end{aligned}$$

From here

$$\begin{aligned}
&\langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n\rangle + \alpha\langle \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle + \\
+ \quad &\alpha\langle \nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n) - \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_1(\bar{x}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle \geq 0
\end{aligned}$$

or taking into account of (22)

$$\begin{aligned}
&\langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n\rangle + \alpha\langle \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle + \\
+ \quad &\alpha^2|\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n) - \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_1(\bar{x}_1^n)|^2 \geq 0.
\end{aligned} \tag{27}$$

We get analogous inequality with respect to variables z_2

$$\begin{aligned}
&\langle \bar{x}_2^n - x_2^n, x_2^{n+1} - \bar{x}_2^n\rangle + \alpha\langle \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n), x_2^{n+1} - \bar{x}_2^n\rangle + \\
+ \quad &\alpha^2|\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n) - \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_2(\bar{x}_2^n)|^2 \geq 0.
\end{aligned} \tag{28}$$

We add systems pair of inequalities (26) and (27),(28)

$$\begin{aligned}
&\langle x_1^{n+1} - x_1^n, x_1^* - x_1^{n+1}\rangle + \langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n\rangle + \\
+ \quad &\langle x_2^{n+1} - x_2^n, x_2^* - x_2^{n+1}\rangle + \langle \bar{x}_2^n - x_2^n, x_2^{n+1} - \bar{x}_2^n\rangle + \\
+ \quad &\alpha\langle \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), x_1^* - \bar{x}_1^n\rangle + \\
+ \quad &\alpha\langle \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n), x_2^* - \bar{x}_2^n\rangle + \\
+ \quad &\alpha^2|\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n) - \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_1(\bar{x}_1^n)|^2 + \\
+ \quad &\alpha^2|\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n) - \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_2(\bar{x}_2^n)|^2 \geq 0.
\end{aligned} \tag{29}$$

Represent (29) in the vector form

$$\begin{aligned}
&(x_1^{n+1} - x_1^n, x_2^{n+1} - x_2^n)\begin{pmatrix} x_1^* - x_1^{n+1} \\ x_2^* - x_2^{n+1} \end{pmatrix} + (\bar{x}_1^n - x_1^n, \bar{x}_2^n - x_2^n)\begin{pmatrix} x_1^{n+1} - \bar{x}_1^n \\ x_2^{n+1} - \bar{x}_2^n \end{pmatrix} + \\
+ \quad &(\nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n))\begin{pmatrix} x_1^* - \bar{x}_1^n \\ x_2^* - \bar{x}_2^n \end{pmatrix} + \\
+ \quad &\alpha^2|\nabla_1 f_1(x_1^n, x_2^n) + \nabla\varphi_1(x_1^n) - \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_1(\bar{x}_1^n)|^2 + \\
+ \quad &\alpha^2|\nabla_2 f_2(x_1^n, x_2^n) + \nabla\varphi_2(x_2^n) - \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) - \nabla\varphi_2(\bar{x}_2^n)|^2 \geq 0.
\end{aligned}$$

Recall notations for vectors introduced previous to $v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $v^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix}$. If we differentiate function $\Phi(v, w) + \varphi(w)$ in w for any v and take the restriction of gradient for this function onto the diagonal of square $\Omega \times \Omega$, (i.e. at $v = w$) then we get from (25)

$$\nabla_2\Phi(v, w)|_{v=w} + \nabla\varphi(w) = \begin{pmatrix} \nabla_1 f_1(z_1, x_2) \\ \nabla_2 f_2(x_1, z_2) \end{pmatrix}_{z_1 = x_1, z_2 = x_2} + \begin{pmatrix} \nabla\varphi_1(z_1) \\ \nabla\varphi_2(z_2) \end{pmatrix}$$

In particular, at $z_1 = \bar{x}_1^n$, $z_2 = \bar{x}_2^n$ we have

$$\nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n) = \left(\begin{array}{c} \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n) \\ \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n) \end{array} \right) \tag{30}$$

In view of entered notations we copy the last inequality in the form

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \\ + & \alpha\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), v^* - \bar{v}^n\rangle + \\ + & \alpha^2|\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) - \nabla_2\Phi(\bar{v}^n, \bar{v}^n) - \nabla\varphi(\bar{v}^n)|^2 \geq 0. \end{array} \tag{31}$$

If operator $\nabla_2\Phi(v, v)$, $\nabla\varphi(v)$ satisfies the Lipschitz condition, then the estimate is correct

$$|\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) - \nabla_2\Phi(\bar{v}^n, \bar{v}^n) - \nabla\varphi(\bar{v}^n)| \leq L|v^n - \bar{v}^n|$$

On the other hand, third addend from (31) is nonpositive by virtue of estimation (13) at $w = \bar{v}^n$. With regard for above inequality (31) takes the one

$$\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + (\alpha L)^2|v^n - \bar{v}^n|^2 \geq 0. \tag{32}$$

By means of identity

$$|v_1 - v_3|^2 = |v_1 - v_2|^2 + 2\langle v_1 - v_2, v_2 - v_3\rangle + |v_2 - v_3|^2, \tag{33}$$

we split of the first two scalar products in (32)

$$\begin{array}{c} |v^{n+1} - v^*|^2 + |v^{n+1} - \bar{v}^n|^2 + |\bar{v}^n - v^n|^2 + \\ +2(\alpha L)^2|\bar{v}^n - v^n)|^2 \leq |v^n - v^*|^2. \end{array}$$

From here, we yield

$$|v^{n+1} - v^*|^2 + |v^{n+1} - \bar{v}^n|^2 + (1 - 2(\alpha L)^2)|\bar{v}^n - v^n|^2 \leq |v^n - v^*|^2. \tag{34}$$

Let us sum up inequality obtained from $n = 0$ to $n = N$

$$|v^{N+1} - v^*|^2 + \sum_{k=0}^{k=N} |v^{k+1} - \bar{v}^k|^2 + d\sum_{k=0}^{k=N} |\bar{v}^k - v^k|^2 \leq |v^0 - v^*|^2,$$

where $d = 1 - 2(\alpha L)^2 > 0$ under the theorem conditions. From the gained inequality the boundedness of trajectory follows

$$|v^{N+1} - v^*|^2 \leq |v^0 - v^*|^2,$$

and the series are convergent $\sum_{k=0}^{\infty} |v^{k+1} - \bar{v}^k|^2 < \infty$, $\sum_{k=0}^{\infty} |\bar{v}^k - v^k|^2 < \infty$. Consequently, values tend to zero $|v^{n+1} - \bar{v}^n|^2 \to 0$, $|\bar{v}^n - v^n|^2 \to 0$, $n \to \infty$. We have too that the sequence v^n is limited. It means, there exists a point v' such that $v^{n_i} \to v'$ as $n_i \to \infty$, and $|v^{n_i+1} - v^{n_i}|^2 \to 0$, $|\bar{v}^{n_i} - v^{n_i}|^2 to 0$.

We put $n = n_i$ in (23) or (24) and passing to the limit as $n_i \to \infty$, we get

$$\begin{aligned} &\langle \nabla_1 f_1(x_1', x_2') + \nabla\varphi_1(x_1'), z_1 - x_1' \rangle \geq 0 \qquad \forall z_1 \in X_1, \\ &\langle \nabla_2 f_2(x_1', x_2') + \nabla\varphi_2(x_2'), z_2 - x_2' \rangle \geq 0 \qquad \forall z_2 \in X_2. \end{aligned}$$

The obtained ratios are similar (18), therefore $x_1', x_2' = x_1^*, x_2^*$, i.e. any limiting point of sequence x_1^n, x_2^n is a solution of problem. The monotonicity condition for decrease of value $|v^n - v^*|$ guarantees the uniqueness of the limit point, i.e. convergence provides $v^n = (x_1^n, x_2^n) \to v^* = (x_1^*, x_2^*)$ as $n \to \infty$. The theorem is proved.

4 Extragradient game methods using Lagrange function

We consider now convex two person non-zero sum game, where each players of the game has, in addition, functional constraints

$$\begin{aligned} &x_1^* \in \text{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) \mid g_1(z_1) \leq 0, \;\; z_1 \in X_1\}, \\ &x_2^* \in \text{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) \mid g_2(z_2) \leq 0, \;\; z_2 \in X_2\}. \end{aligned} \qquad (35)$$

Each of participants of the game decides a convex programming problem in an own variable at fixed values parameters. We introduce the Lagrange functions for each of the players. These functions depend on parameters $v = (x_1, x_2)$

$$\begin{aligned} &\mathcal{L}_1(z_1, x_2, p_1) = f_1(z_1, x_2) + \varphi_1(z_1) + \langle p_1, g_1(z_1)\rangle \qquad \forall z_1 \in X_1, \;\; p_1 \geq 0, \\ &\mathcal{L}_2(x_1, z_2, p_2) = f_2(x_1, z_2) + \varphi_2(z_2) + \langle p_2, g_2(z_2)\rangle \qquad \forall z_2 \in X_2, \;\; p_2 \geq 0. \end{aligned} \qquad (36)$$

We assume that under equilibrium conditions, i.e. at $x_1 = x_1^*, x_2 = x_2^*$ points x_1^*, p_1^* and x_2^*, p_2^* are saddle points for Lagrange functions $\mathcal{L}_1(z_1, x_2^*, p_1)$ and $\mathcal{L}_2(x_1^*, z_2, p_2)$. The latter means that inequality system is held

$$\mathcal{L}_1(x_1^*, x_2^*, p_1) \leq \mathcal{L}_1(x_1^*, x_2^*, p_1^*) \leq \mathcal{L}_1(z_1, x_2^*, p_1^*) \qquad \forall z_1 \geq 0, \;\; \forall p_1 \geq 0, \qquad (37)$$

$$\mathcal{L}_2(x_1^*, x_2^*, p_2) \leq \mathcal{L}_2(x_1^*, x_2^*, p_2^*) \leq \mathcal{L}_2(x_1^*, z_2, p_2^*) \qquad \forall z_2 \geq 0, \;\; \forall p_2 \geq 0. \qquad (38)$$

We rewrite inequalities (37), (38) in the form of the system of problems

$$\begin{aligned} x_1^* &\in \text{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) + \langle p_1^*, g_1(z_1)\rangle \mid \forall z_1 \in X_1\}, \\ p_1^* &\in \text{Argmax}\{\langle p_1, g_1(x_1^*)\rangle \mid p_1 \geq 0\}, \qquad (39) \\ x_2^* &\in \text{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) + \langle p_2^*, g_2(z_2)\rangle \mid \forall z_2 \in X_2\}, \\ p_2^* &\in \text{Argmax}\{\langle p_2, g_2(x_2^*)\rangle \mid p_2 \geq 0\} \qquad (40) \end{aligned}$$

or in the form of variational inequalities

$$\begin{aligned} &\langle \nabla_1 f_1(x_1^*, x_2^*) + \nabla \varphi_1(x_1^*) + \nabla g_1^\top(x_1^*) p_1^*, z_1 - x_1^* \rangle \geq 0 \quad \forall z_1 \in X_1, \\ -&\langle g_1(x_1^*), p_1 - p_1^* \rangle \geq 0 \quad \forall p_1 \geq 0, \\ &\langle \nabla_2 f_2(x_1^*, x_2^*) + \nabla \varphi_2(x_2^*) + \nabla g_2^\top(x_2^*) p_2^*, z_2 - x_2^* \rangle \geq 0 \quad \forall z_2 \in X_2, \\ -&\langle g_2(x_2^*), p_2 - p_2^* \rangle \geq 0 \quad \forall p_2 \geq 0, \end{aligned} \tag{41}$$

where $\nabla g_1^\top(x_1)$, $\nabla g_2^\top(x_2)$ are $m_1 \times n$ and $m_2 \times n$ matrices, and $\nabla g_{1,i}(x_1)$, $\nabla g_{2,j}(x_2)$, $i = 1, 2, \ldots, m_1$, $j = 1, 2, \ldots, m_2$ are vector-lines.

We differentiate the Lagrange functions (36) in $w = (z_1, z_2)$

$$\begin{aligned} \nabla_1 \mathcal{L}_1(z_1, x_2, p_1) = \nabla_1 f_1(z_1, x_2) + \nabla \varphi_1(z_1) + \nabla g_1^\top(z_1) p_1, \\ \nabla_2 \mathcal{L}_2(x_1, z_2, p_2) = \nabla_2 f_2(x_1, z_2) + \nabla \varphi_2(z_2) + \nabla g_2^\top(z_2) p_2, \end{aligned} \tag{42}$$

and present the system of variational inequalities (41) in the equivalent form of operator equations

$$\begin{aligned} x_1^* = \pi_{X_1}(x_1^* - \alpha(\nabla_1 \mathcal{L}_1(x_1^*, x_2^*, p_1^*)), \qquad p_1^* = \pi_+(p_1^* + \alpha g_1(x_1^*)), \\ x_2^* = \pi_{X_2}(x_2^* - \alpha(\nabla_2 \mathcal{L}_2(x_1^*, x_2^*, p_2^*)), \qquad p_2^* = \pi_+(p_2^* + \alpha g_2(x_2^*)), \end{aligned} \tag{43}$$

where $\pi_+(\ldots)$ is projection operator of some vector into the positive orthant R^n_+, $\alpha > 0$ is a parameter like steplength.

For the solution of system (43) we use the extragradient method with respect to primal and dual variables, Antipin 1997, Antipin 2000, Antipin 2001A. The method includes two half steps:

the first half-step

$$\begin{aligned} \bar{p}_1^n = \pi_+(p_1^n + \alpha g_1(x_1^n)), \qquad \bar{x}_1^n = \pi_{X_1}(x_1^n - \alpha \nabla_1 \mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n)), \\ \bar{p}_2^n = \pi_+(p_2^n + \alpha g_2(x_2^n)), \qquad \bar{x}_2^n = \pi_{X_2}(x_2^n - \alpha \nabla_2 \mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n)), \end{aligned} \tag{44}$$

and second half-step

$$\begin{aligned} p_1^{n+1} = \pi_+(p_1^n + \alpha g_1(\bar{x}_1^n)), \qquad x_1^{n+1} = \pi_{X_1}(x_1^n - \alpha \nabla_1 \mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)), \\ p_2^{n+1} = \pi_+(p_2^n + \alpha g_2(\bar{x}_2^n)), \qquad x_2^{n+1} = \pi_{X_2}(x_2^n - \alpha \nabla_2 \mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)). \end{aligned} \tag{45}$$

The steplength α in (44), (45) is determined from a interval $0 < \alpha < \alpha_0$, where right-hand side of this segment will be determined later.

For the justification of correctness of selecting out parameter α we receive evaluations of deviations for vectors $\bar{x}_1^n, \bar{x}_2^n$ and x_1^{n+1}, x_2^{n+1} and $\bar{p}_1^n, \bar{p}_2^n$ and p_1^{n+1}, p_2^{n+1} from (44), (45)

$$\begin{aligned} &|\bar{p}_1^n - p_1^{n+1}| \leq \alpha |g_1(x_1^n) - g_1(\bar{x}_1^n)|, \\ &|\bar{p}_2^n - p_2^{n+1}| \leq \alpha |g_2(x_2^n) - g_2(\bar{x}_2^n)|, \\ &|\bar{x}_1^n - x_1^{n+1}| \leq \alpha |\nabla_1 \mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1 \mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)|, \\ &|\bar{x}_2^n - x_2^{n+1}| \leq \alpha |\nabla_2 \mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n) - \nabla_2 \mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)|. \end{aligned} \tag{46}$$

We rewrite process (44), (45) in the form of variational inequalities. According with the definition of projection operator we present equations from (44) in the kind

$$\begin{array}{ll} \langle \bar{p}_1^n - p_1^n - \alpha g_1(x_1^n), p_1 - \bar{p}_1^n \rangle \geq 0 & \forall p_1 \geq 0, \\ \langle \bar{p}_2^n - p_2^n - \alpha g_2(x_2^n), p_2 - \bar{p}_2^n \rangle \geq 0 & \forall p_2 \geq 0 \end{array} \tag{47}$$

and

$$\begin{array}{ll} \langle \bar{x}_1^n - x_1^n + \alpha \nabla_1 \mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n), z_1 - \bar{x}_1^n \rangle \geq 0 & \forall z_1 \in X_1, \\ \langle \bar{x}_2^n - x_2^n + \alpha \nabla_2 \mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n), z_2 - \bar{x}_2^n \rangle \geq 0 & \forall z_2 \in X_2. \end{array} \tag{48}$$

We rewrite equations (45) as

$$\begin{array}{ll} \langle p_1^{n+1} - p_1^n - \alpha g_1(\bar{x}_1^n), p_1 - p_1^{n+1} \rangle \geq 0 & \forall p_1 \geq 0, \\ \langle p_2^{n+1} - p_2^n - \alpha g_2(\bar{x}_2^n), p_2 - p_2^{n+1} \rangle \geq 0 & \forall p_2 \geq 0 \end{array} \tag{49}$$

and

$$\begin{array}{ll} \langle x_1^{n+1} - x_1^n + \alpha \nabla_1 \mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), z_1 - x_1^{n+1} \rangle \geq 0 & \forall z_1 \in X_1, \\ \langle x_2^{n+1} - x_2^n + \alpha \nabla_2 \mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n), z_2 - x_2^{n+1} \rangle \geq 0 & \forall z_2 \in X_2. \end{array} \tag{50}$$

In theorem 1 we used inequality (13), which is the corollary of monotonicity property for gradient-restriction $\nabla_2\Phi(v, v) + \nabla\varphi(v)$. In the considered situation we generalize this inequality for the case when functional constraints of equilibrium problem are taking into account by means of Lagrange functions. To this end we put $z_1 = \bar{x}_1^n$, $z_2 = \bar{x}_2^n$ in the first and third inequalities (41)

$$\begin{array}{l} \langle \nabla_1 f_1(x_1^*, x_2^*) + \nabla\varphi_1(x_1^*) + \nabla g_1^\top(x_1^*)p_1^*, \bar{x}_1^n - x_1^* \rangle \geq 0, \\ \langle \nabla_2 f_2(x_1^*, x_2^*) + \nabla\varphi_2(x_2^*) + \nabla g_2^\top(x_2^*)p_2^*, \bar{x}_2^n - x_2^* \rangle \geq 0. \end{array}$$

From here

$$\begin{array}{l} \langle \nabla_1 f_1(x_1^*, x_2^*) + \nabla\varphi_1(x_1^*), \bar{x}_1^n - x_1^* \rangle + \langle p_1^*, \nabla g_1(x_1^*)(\bar{x}_1^n - x_1^*) \rangle \geq 0, \\ \langle \nabla_2 f_2(x_1^*, x_2^*) + \nabla\varphi_2(x_2^*), \bar{x}_2^n - x_2^* \rangle + \langle p_2^*, \nabla g_2(x_2^*)(\bar{x}_2^n - x_2^*) \rangle \geq 0. \end{array}$$

Using the convexity of vector functions $g_1(x_1)$, $g_2(x_2)$, we add both inequalities

$$\begin{array}{rl} & \langle \nabla_1 f_1(x_1^*, x_2^*) + \nabla\varphi_1(x_1^*), \bar{x}_1^n - x_1^* \rangle + \langle \nabla_2 f_2(x_1^*, x_2^*) + \nabla\varphi_2(x_2^*), \bar{x}_2^n - x_2^* \rangle + \\ + & \langle p_1^*, g_1(\bar{x}_1^n) - g_1(x_1^*) \rangle + \langle p_2^*, g_2(\bar{x}_2^n) - g_2(x_2^*) \rangle \geq 0. \end{array} \tag{51}$$

We use the obtained estimate in proving the following

Theorem 2 *Suppose that a solution set of game* (35) *is non-empty, Lagrange function of each player has got a saddle point, normalized function of this game* $\Phi(v, w)+$ $+\varphi(w)$ *is positive semidefinite and convex in* w *for any* v*, its gradient-restriction* $\nabla_2\Phi(v, w)|_{v=w} + \nabla\varphi(w)$ *satisfies the Lipschitz condition with constant* L_1*, vector and matrix functions* $g(w)$*,* $\nabla g^\top(w)$ *satisfy the Lipschitz conditions with constants* L_3 *u* L_2*, sequence* $\bar{p}^n \leq C$ *is limited for all* n*,* $\Omega \subseteq R^n$ *is convex closed set. Then, the sequence* x_1^n, x_2^n*,* p_1^n, p_2^n*, generated by method* (44), (45) *with steplength* α *chosen from condition* $0 < \alpha < 1/(\sqrt{2(L_1^2 + (CL_2)^2) + L_3^2}$ *converges to a game solution, i.e. m.e.* $x_1^n, x_2^n \to x_1^*, x_2^*$*,* $p_1^n, p_2^n \to p_1^*, p_2^*$ *as* $n \to \infty$ *monotonically in the norm.*

PROOF. We put $z_1 = x_1^*$, $z_2 = x_2^*$ in (50), then

$$\begin{aligned} &\langle x_1^{n+1} - x_1^n + \alpha\nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), x_1^* - x_1^{n+1}\rangle \geq 0, \\ &\langle x_2^{n+1} - x_2^n + \alpha\nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n), x_2^* - x_2^{n+1}\rangle \geq 0. \end{aligned} \tag{52}$$

We set $z_1 = x_1^{n+1}$, $z_2 = x_2^{n+1}$ in (48)

$$\begin{aligned} &\langle \bar{x}_1^n - x_1^n + \alpha\nabla_1\mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle \geq 0, \\ &\langle \bar{x}_2^n - x_2^n + \alpha\nabla_2\mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n), x_2^{n+1} - \bar{x}_2^n\rangle \geq 0. \end{aligned} \tag{53}$$

Hence

$$\begin{aligned} &\langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n\rangle + \alpha\langle\nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle + \\ + \quad &\alpha\langle\nabla_1\mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle \geq 0, \end{aligned}$$

taking into account (46), we have

$$\begin{aligned} &\langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n\rangle + \alpha\langle\nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), x_1^{n+1} - \bar{x}_1^n\rangle + \\ + \quad &\alpha^2|\nabla_1\mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)|^2 \geq 0. \end{aligned} \tag{54}$$

We receive the similar inequality with respect to variable z_2

$$\begin{aligned} &\langle \bar{x}_2^n - x_2^n, x_2^{n+1} - \bar{x}_2^n\rangle + \alpha\langle\nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n), x_2^{n+1} - \bar{x}_2^n\rangle + \\ + \quad &\alpha^2|\nabla_2\mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n) - \nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)|^2 \geq 0. \end{aligned} \tag{55}$$

Add systems of pairs of inequalities (52) and (54), (55)

$$\begin{aligned} &\langle x_1^{n+1} - x_1^n, x_1^* - x_1^{n+1}\rangle + \langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n\rangle + \\ + \quad &\langle x_2^{n+1} - x_2^n, x_2^* - x_2^{n+1}\rangle + \langle \bar{x}_2^n - x_2^n, x_2^{n+1} - \bar{x}_2^n\rangle + \\ + \quad &\alpha\langle\nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), x_1^* - \bar{x}_1^n\rangle + \alpha\langle\nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n), x_2^* - \bar{x}_2^n\rangle + \\ + \quad &\alpha^2|\nabla_1\mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)|^2 + \\ + \quad &\alpha^2|\nabla_2\mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n) - \nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)|^2 \geq 0. \end{aligned}$$

In view of convexity $g_1(x_1)$, $g_2(x_2)$ we estimate separately the fifth and sixth term in obtained inequality

$$\begin{aligned} &\langle\nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n), x_1^* - \bar{x}_1^n\rangle + \langle\nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n), x_2^* - \bar{x}_2^n\rangle = \\ = \quad &\langle\nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), x_1^* - \bar{x}_1^n\rangle + \langle\nabla g_1^\top(\bar{x}_1^n)\bar{p}_1^n, x_1^* - \bar{x}_1^n)\rangle + \\ + \quad &\langle\nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n), x_2^* - \bar{x}_2^n\rangle + \langle\nabla g_2^\top(\bar{x}_2^n)\bar{p}_2^n, x_2^* - \bar{x}_2^n\rangle \leq \\ \leq \quad &\langle\nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), x_1^* - \bar{x}_1^n\rangle + \langle\bar{p}_1^n, g_1(x_1^*) - g_1(\bar{x}_1^n)\rangle + \\ + \quad &\langle\nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n), x_2^* - \bar{x}_2^n\rangle + \langle\bar{p}_2^n, g_2(x_2^*) - g_2(\bar{x}_2^n)\rangle. \end{aligned}$$

Then

$$
\begin{aligned}
& \langle x_1^{n+1} - x_1^n, x_1^* - x_1^{n+1} \rangle + \langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n \rangle + \\
+ \quad & \langle x_2^{n+1} - x_2^n, x_2^* - x_2^{n+1} \rangle + \langle \bar{x}_2^n - x_2^n, x_2^{n+1} - \bar{x}_2^n \rangle + \\
+ \quad & \alpha(\langle \nabla_1 f_1(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_1(\bar{x}_1^n), x_1^* - \bar{x}_1^n \rangle + \langle \bar{p}_1^n, g_1(x_1^*) - g_1(\bar{x}_1^n) \rangle) + \\
+ \quad & \alpha(\langle \nabla_2 f_2(\bar{x}_1^n, \bar{x}_2^n) + \nabla\varphi_2(\bar{x}_2^n), x_2^* - \bar{x}_2^n \rangle + \langle \bar{p}_2^n, g_2(x_2^*) - g_2(\bar{x}_2^n) \rangle + \\
+ \quad & \alpha^2 |\nabla_1 \mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1 \mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)|^2 + \\
+ \quad & \alpha^2 |\nabla_2 \mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n) - \nabla_2 \mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)|^2 \geq 0.
\end{aligned} \tag{56}
$$

Using notations (30), we add inequality (51) and (56), then

$$
\begin{aligned}
& \langle x_1^{n+1} - x_1^n, x_1^* - x_1^{n+1} \rangle + \langle \bar{x}_1^n - x_1^n, x_1^{n+1} - \bar{x}_1^n \rangle + \\
+ \quad & \langle x_2^{n+1} - x_2^n, x_2^* - x_2^{n+1} \rangle + \langle \bar{x}_2^n - x_2^n, x_2^{n+1} - \bar{x}_2^n \rangle + \\
+ \quad & \langle \nabla_2 \Phi(\bar{v}_1^n, \bar{v}_2^n) + \nabla\varphi(\bar{v}_1^n) - \nabla_2 \Phi(v_1^*, v_2^*) - \nabla\varphi(v_1^*), v_1^* - \bar{v}_1^n \rangle + \\
+ \quad & \langle \bar{p}_1^n - p_1^*, g_1(x_1^*) - g_1(\bar{x}_1^n) \rangle + \langle \bar{p}_2^n - p_2^*, g_2(x_2^*) - g_2(\bar{x}_2^n) \rangle + \\
+ \quad & \alpha^2 |\nabla_1 \mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1 \mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)|^2 + \\
+ \quad & \alpha^2 |\nabla_2 \mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n) - \nabla_2 \mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)|^2 \geq 0.
\end{aligned} \tag{57}
$$

To the obtained inequality we return back later and now consider inequalities (47) and (49). Let us put $p_1 = p_1^*$, $p_2 = p_2^*$ in (49)

$$
\begin{aligned}
& \langle p_1^{n+1} - p_1^n, p_1^* - p_1^{n+1} \rangle - \alpha \langle g_1(\bar{x}_1^n), p_1^* - p_1^{n+1} \rangle \geq 0, \\
& \langle p_2^{n+1} - p_2^n, p_2^* - p_2^{n+1} \rangle - \alpha \langle g_2(\bar{x}_2^n), p_2^* - p_2^{n+1} \rangle \geq 0
\end{aligned} \tag{58}
$$

and $p_1 = p_1^{n+1}, \; p_2 = p_2^{n+1}$ in (47), then

$$
\begin{aligned}
& \langle \bar{p}_1^n - p_1^n, p_1^{n+1} - \bar{p}_1^n \rangle + \alpha \langle g_1(\bar{x}_1^n) - g_1(x_1^n), p_1^{n+1} - \bar{p}_1^n \rangle - \\
- \quad & \alpha \langle g_1(\bar{x}_1^n), p_1^{n+1} - \bar{p}_1^n \rangle \geq 0, \\
& \langle \bar{p}_2^n - p_2^n, p_2^{n+1} - \bar{p}_2^n \rangle + \alpha \langle g_2(\bar{x}_2^n) - g_2(x_2^n), p_2^{n+1} - \bar{p}_2^n \rangle - \\
- \quad & \alpha \langle g_2(\bar{x}_2^n), p_2^{n+1} - \bar{p}_2^n \rangle \geq 0.
\end{aligned} \tag{59}
$$

We combine both inequalities (58) and (59)

$$
\begin{aligned}
& \langle p_1^{n+1} - p_1^n, p_1^* - p_1^{n+1} \rangle + \langle p_2^{n+1} - p_2^n, p_2^* - p_2^{n+1} \rangle + \\
+ \quad & \langle \bar{p}_1^n - p_1^n, p_1^{n+1} - \bar{p}_1^n \rangle + \langle \bar{p}_2^n - p_2^n, p_2^{n+1} - \bar{p}_2^n \rangle + \\
+ \quad & \alpha \langle g_1(\bar{x}_1^n) - g_1(x_1^n), p_1^{n+1} - \bar{p}_1^n \rangle + \alpha \langle g_2(\bar{x}_2^n) - g_2(x_2^n), p_2^{n+1} - \bar{p}_2^n \rangle - \\
- \quad & \alpha \langle g_1(\bar{x}_1^n), p_1^* - \bar{p}_1^n \rangle - \alpha \langle g_2(\bar{x}_2^n), p_2^* - \bar{p}_2^n \rangle \geq 0.
\end{aligned}
$$

Using (46), rewrite the last inequality in the kind

$$\begin{array}{rl} & \langle p_1^{n+1}-p_1^n, p_1^*-p_1^{n+1}\rangle + \langle p_2^{n+1}-p_2^n, p_2^*-p_2^{n+1}\rangle + \\ + & \langle \bar p_1^n - p_1^n, p_1^{n+1}-\bar p_1^n\rangle + \langle \bar p_2^n - p_2^n, p_2^{n+1}-\bar p_2^n\rangle + \\ + & \alpha^2|g_1(\bar x_1^n)-g_1(x_1^n)|^2 + \alpha^2|g_2(\bar x_2^n)-g_2(x_2^n)|^2 - \\ - & \alpha\langle g_1(\bar x_1^n), p_1^*-\bar p_1^n\rangle - \alpha\langle g_2(\bar x_2^n), p_2^*-\bar p_2^n\rangle \ge 0. \end{array} \tag{60}$$

We put $p_1=\bar p_1^n$, $p_2=\bar p_2^n$ in second and fourth inequalities (41) and add both ones to (60), then

$$\begin{array}{rl} & \langle p_1^{n+1}-p_1^n, p_1^*-p_1^{n+1}\rangle + \langle p_2^{n+1}-p_2^n, p_2^*-p_2^{n+1}\rangle + \\ + & \langle \bar p_1^n - p_1^n, p_1^{n+1}-\bar p_1^n\rangle + \langle \bar p_2^n - p_2^n, p_2^{n+1}-\bar p_2^n\rangle + \\ + & \alpha^2|g_1(\bar x_1^n)-g_1(x_1^n)|^2 + \alpha^2|g_2(\bar x_2^n)-g_2(x_2^n)|^2 + \\ + & \alpha\langle g_1(x_1^*)-g_1(\bar x_1^n), p_1^*-\bar p_1^n\rangle + \alpha\langle g_2(x_2^*)-g_2(\bar x_2^n), p_2^*-\bar p_2^n\rangle \ge 0. \end{array} \tag{61}$$

At last, we combine inequalities (57) and (61) and take into account the monotonicity condition (11) of the operator $\nabla_2\Phi(v,v)+\nabla\varphi(v)$, then

$$\begin{array}{rl} & \langle x_1^{n+1}-x_1^n, x_1^*-x_1^{n+1}\rangle + \langle \bar x_1^n - x_1^n, x_1^{n+1}-\bar x_1^n\rangle + \\ + & \langle x_2^{n+1}-x_2^n, x_2^*-x_2^{n+1}\rangle + \langle \bar x_2^n - x_2^n, x_2^{n+1}-\bar x_2^n\rangle + \\ + & \langle p_1^{n+1}-p_1^n, p_1^*-p_1^{n+1}\rangle + \langle p_2^{n+1}-p_2^n, p_2^*-p_2^{n+1}\rangle + \\ + & \langle \bar p_1^n - p_1^n, p_1^{n+1}-\bar p_1^n\rangle + \langle \bar p_2^n - p_2^n, p_2^{n+1}-\bar p_2^n\rangle + \\ + & \alpha^2|\nabla_1\mathcal{L}_1(x_1^n,x_2^n,\bar p_1^n)-\nabla_1\mathcal{L}_1(\bar x_1^n,\bar x_2^n,\bar p_1^n)|^2 + \\ + & \alpha^2|\nabla_2\mathcal{L}_2(x_1^n,x_2^n,\bar p_2^n)-\nabla_2\mathcal{L}_2(\bar x_1^n,\bar x_2^n,\bar p_2^n)|^2 + \\ + & \alpha^2|g_1(\bar x_1^n)-g_1(x_1^n)|^2 + \alpha^2|g_2(\bar x_2^n)-g_2(x_2^n)|^2 \ge 0. \end{array} \tag{62}$$

Further, following logic to the theorem 1, we scalarized the obtained inequality and pass to the variables $v=(x_1,x_2)^\top$, $p=(p_1,p_2)^\top$. To this end we consider three last addends in (62) in more details. We expand convolution (25) with the help of scalarization of two Lagrange functions (36)

$$\begin{aligned} \mathcal{L}(v,w,p) &= \Phi(v,w)+\varphi(w)+\langle p, g(w)\rangle = \mathcal{L}_1(z_1,x_2,p_1)+\mathcal{L}_2(x_1,z_2,p_2) = \\ &= f_1(z_1,x_2)+\varphi_1(z_1)+\langle p_1, g_1(z_1)\rangle + f_2(x_1,z_2)+\varphi_2(z_2)+\langle p_2, g_2(z_2)\rangle. \end{aligned}$$

Differentiating function $\mathcal{L}(v,w,p)$ in w, we consider gradient-restriction of this function $\nabla_2\mathcal{L}(v,w,p)|_{v=w}$ on the diagonal of square $\Omega\times\Omega$

$$\nabla_2\mathcal{L}(v,w,p)|_{v=w} = \nabla_2\Phi(v,w)|_{v=w}+\nabla\varphi(w)+\nabla g^\top(w)p = \begin{pmatrix} \nabla_1\mathcal{L}_1(z_1,x_2,p_1) \\ \nabla_2\mathcal{L}_2(x_1,z_2,p_2) \end{pmatrix}_{\substack{z_1=x_1,\\ z_2=x_2}}$$

In particular, at $z_1 = \bar{x}_1^n$, $z_2 = \bar{x}_2^n$, $p_1 = \bar{p}_1^n$, $p_2 = \bar{p}_2^n$ we have

$$\nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n) + \nabla g^\top(\bar{v}^n)\bar{p}^n = \begin{pmatrix} \nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n) \\ \nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n) \end{pmatrix}$$

Using the conditions

$$\begin{aligned}
&|\nabla_2\Phi(v+h, v+h) + \nabla\varphi(v+h) - \nabla_2\Phi(\bar{v}, \bar{v}) - \nabla\varphi(\bar{v})| \le L_1|h|, \\
&|\nabla g^\top(v+h) - \nabla g^\top(\bar{v})| \le L_2|h|, \\
&|g(v+h) - g(v)| \le L_3|h|, \quad |\bar{p}^n| \le C, \quad n \to \infty.
\end{aligned}$$

estimate three last term from (62)

$$\begin{aligned}
& |\nabla_1\mathcal{L}_1(x_1^n, x_2^n, \bar{p}_1^n) - \nabla_1\mathcal{L}_1(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_1^n)|^2 + \\
+ \;& |\nabla_2\mathcal{L}_2(x_1^n, x_2^n, \bar{p}_2^n) - \nabla_2\mathcal{L}_2(\bar{x}_1^n, \bar{x}_2^n, \bar{p}_2^n)|^2 = \\
= \;& |\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) - \nabla_2\Phi(\bar{v}^n, \bar{v}^n) - \nabla\varphi(\bar{v}^n) + \\
+ \;& (\nabla g^\top(v^n) - \nabla g^\top(\bar{v}^n))\bar{p}^n|^2 \le \\
\le \;& L_1^2|v^n - \bar{v}^n|^2 + (CL_2)^2|v^n - \bar{v}^n|^2 = (L_1^2 + (CL_2)^2)|v^n - \bar{v}^n|^2
\end{aligned}$$

and

$$|g(v^n) - g(\bar{v}^n)|^2 = |g_1(\bar{x}_1^n) - g_1(x_1^n)|^2 + |g_2(\bar{x}_2^n) - g_2(x_2^n)|^2 = L_3^2|v^n - \bar{v}^n|^2.$$

In view of said above we present (62) as

$$\begin{aligned}
& \langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \\
+ \;& \langle p^{n+1} - p^n, p^* - p^{n+1}\rangle + \langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle + \\
+ \;& \alpha^2(L_1^2 + (CL_2)^2)|v^n - \bar{v}^n| + \alpha^2 L_3^2|v^n - \bar{v}^n|^2 \ge 0.
\end{aligned}$$

Applying identity (33), we bring the obtained inequality to the kind

$$\begin{aligned}
|v^{n+1} - v^*|^2 &+ |v^{n+1} - \bar{v}^n|^2 + d|\bar{v}^n - v^n|^2 + |p^{n+1} - p^*|^2 + |p^{n+1} - \bar{p}^n|^2 + \\
&+ |\bar{p}^n - p^n|^2 \le |v^n - v^*|^2 + |p^n - p^*|^2,
\end{aligned} \tag{63}$$

where $d = 1 - 2\alpha^2(L_1^2 + (CL_2)^2) + L_3^2) \ge 0$ is under conditions of the theorem, as

$$0 < \alpha < \frac{1}{(\sqrt{2(L_1^2 + (CL_2)^2) + L_3^2)}}.$$

The inequality (63) is exact analog of (34), therefore if we continue reasoning of the theorem 1, then we easily establish the monotone convergence of sequence $x_1^n, x_2^n \to x_1^*, x_2^*$, $p_1^n, p_2^n \to p_1^*, p_2^*$ as $n \to \infty$ to game solution of the problem. The theorem is proved.

5 Game problems with coupled constraints

Let the objective functions $f_1(x_1, x_2) + \varphi_1(x_1), f_2(x_1, x_2) + \varphi_2(x_2)$ and the functional vectorial constraint $g(x_1, x_2) \le 0$, where $g(x_1, x_2) = (g_1(x_1, x_2), \ldots, g_m(x_1, x_2))$ be defined on the product space $R_1^{n_1} \times R_2^{n_2}$. It is supposed that functions $f_1(z_1, x_2) + \varphi(z_1)$, $f_2(x_1, z_2) + \varphi(z_2)$ are continuous and convex on the own variable. The constraint functions $g(x_1, x_2)$ are convex in variables x_1 and x_2 separately and, generally speaking, are not convex in these variables jointly, for example, it can be a saddle-type function $g(x_1, x_2) = \langle x_1, x_2 \rangle$.

We consider the following extreme map. Let $G = \{x_1, x_2 \mid g(x_1, x_2) \le 0,\ \ x_1 \in \in X_1,\ \ x_2 \in X_2\}$ be a admissible set. Having taken a certain point $x = (x_1, x_2) \in G$ in this set we take two cross-sections of a kind $G_1(x) = \{z_1 \in X_1 \mid (z_1, x_2) \in G\}$ and $G_2(x) = \{z_2 \in X_2 \mid (x_1, z_2) \in G\}$. Assuming that the objective functions are convex in the own variables for any values of other variables, we consider a pair of optimization problems each of them is on its cross-section

$$\begin{aligned} y_1(x_2) &\in \operatorname{Argmin}\{f_1(z_1, x_2) + \varphi_1(z_1) \mid g(z_1, x_2) \le 0,\ \ z_1 \in X_1\}, \\ y_2(x_1) &\in \operatorname{Argmin}\{f_2(x_1, z_2) + \varphi_2(z_2) \mid g(x_1, z_2) \le 0,\ \ z_2 \in X_2\}. \end{aligned} \tag{64}$$

The system of problems (64) determines the operator $Y(x) = y_1(x_2) \times y_2(x_1)$, which maps any point $x = (x_1, x_2) \in G$ into some convex closed subset from $\pi_1 G \times \pi_2 G$, where $\pi_1 G$ is a projection of set G onto space of variable x_1 and $\pi_2 G$ is a projection of a set G onto space of a variable x_2.

The subset $Y(x)$ represents a direct product of optimal sets of problems (64). If functions $f_1(z_1, x_2) + \varphi(z_1)$, $f_2(x_1, z_2) + \varphi(z_2)$ are continuous and convex in the own variables, and $X_i, i = 1, 2$ are convex compact sets, then the map has a fixed point $x^* = (x_1^*, x_2^*)$, Aubin, Frankowska 1990. It is a point satisfies a system of extreme inclusions

$$\begin{aligned} x_1^* &\in \operatorname{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) \mid g(z_1, x_2^*) \le 0,\ \ z_1 \in X_1\}, \\ x_2^* &\in \operatorname{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) \mid g(x_1^*, z_2) \le 0,\ \ z_2 \in X_2\}. \end{aligned} \tag{65}$$

The problem (65) represents by itself a two-person game with coupled constraints dependent on parameter x_1, x_2. For certain values of parameters each of the game participants try to solve the convex programming problem in own variables. Both problems of this system have, actually, the same constraint but with respect to different variables: the first problem in variable x_1, the second one in variable x_2. In this game situation both players are hardly coupled both objective functions and constraints. The level of independence of players in this situations not high, therefore it is natural to consider the problem in scalarized form. To this end we enter two normalized functions of a kind

$$\begin{aligned} \Phi(v, w) + \varphi(w) &= f_1(z_1, x_2) + \varphi_1(z_1) + f_2(x_1, z_2) + \varphi_2(z_2), \\ G(v, w) &= g(z_1, x_2) + g(x_1, z_2), \end{aligned} \tag{66}$$

where $w = (z_1, z_2)$, $v = (x_1, x_2)$, $v, w \in \Omega = X_1 \times X_2$. In terms of new variables problem(65) can be presented in the form

$$v^* \in \text{Argmin}\{\Phi(v^*, w) + \varphi(w) \mid G(v^*, w) \le 0, \;\; w \in \Omega\} \tag{67}$$

or that is the same

$$\Phi(v^*, v^*) + \varphi(v^*) \le \Phi(v^*, w) + \varphi(w), \qquad G(v^*, w) \le 0 \qquad \forall w \in \Omega. \tag{68}$$

Uneasy to be convinced of equivalence of problems (65) and (67) or (68). Really, we present (68) as

$$f_1(x_1^*, x_2^*) + \varphi_1(x_1^*) + f_2(x_1^*, x_2^*) + \varphi_2(x_2^*) \le f_1(z_1, x_2^*) + \varphi_1(z_1) + f_2(x_1^*, z_2) + \varphi_2(z_2),$$

$$g(z_1, x_2^*) + g(x_1^*, z_2) \le 0, \qquad z_1 \in X_1, \;\; z_2 \in X_2.$$

The objective function and functional constraint in this problem have the separable structure. This circumstance used to split the problem and to reduce it to two-person game. To this end we introduce the Lagrange function

$$\begin{aligned} \mathcal{L}(x_1, x_2, z_1, z_2, \lambda) &= f_1(z_1, x_2) + \varphi_1(z_1) + f_2(x_1, z_2) + \varphi_2(z_2) + \\ &+ \langle \lambda, g(z_1, x_2) + g(x_1, z_2) \rangle, \end{aligned} \tag{69}$$

where $x_1 \in X_1$, $x_2 \in X_2$, $z_1 \in X_1$, $z_2 \in X_2$, $\lambda \ge 0$. Assuming that the optimization problem from (68) subject to constraint qualification, it is possible to assert that x_1^*, x_2^*, λ^* is saddle point for Lagrange function in the state of equilibrium, that is

$$\mathcal{L}(x_1^*, x_2^*, x_1^*, x_2^*, \lambda) \le \mathcal{L}(x_1^*, x_2^*, x_1^*, x_2^*, \lambda^*) \le \mathcal{L}(x_1^*, x_2^*, z_1, z_2, \lambda^*) \tag{70}$$

for all $z_1 \in X_1$, $z_2 \in X_2$, $\lambda \ge 0$. The right-hand side of inequality for this system represents by itself the optimization problem for separable function on square $X_1 \times X_2$. By virtue of separability of objective function in z_1 and z_2 and block structure of constraints the problem is splitted on two independent subtasks, each of them is determined in own space, namely:

$$\begin{aligned} f_1(x_1^*, x_2^*) + \varphi_1(x_1^*) + \langle \lambda^*, g(x_1^*, x_2^*) \rangle &\le f_1(z_1, x_2^*) + \varphi_1(z_1) + \langle \lambda^*, g(z_1, x_2^*) \rangle, \quad z_1 \in X_1, \\ f_2(x_1^*, x_2^*) + \varphi_2(x_2^*) + \langle \lambda^*, g(x_1^*, x_2^*) \rangle &\le f_2(x_1^*, z_2) + \varphi_2(z_2) + \langle \lambda^*, g(x_1^*, z_2) \rangle, \quad z_2 \in X_2. \end{aligned}$$

Obviously that these problems can be rewritten as

$$\begin{aligned} f_1(x_1^*, x_2^*) + \varphi_1(x_1^*) &\le f_1(z_1, x_2^*) + \varphi_1(z_1) + \langle \lambda^*, g(z_1, x_2^*) - g(x_1^*, x_2^*) \rangle, \qquad z_1 \in X_1, \\ f_2(x_1^*, x_2^*) + \varphi_2(x_2^*) &\le f_2(x_1^*, z_2) + \varphi_2(z_2) + \langle \lambda^*, g(x_1^*, z_2) - g(x_1^*, x_2^*) \rangle, \qquad z_2 \in X_2. \end{aligned}$$

Or

$$\begin{aligned} x_1^* &\in \text{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) \mid \langle \lambda^*, g(z_1, x_2^*) - g(x_1^*, x_2^*) \rangle \le 0, \quad z_1 \in X_1\}, \\ x_2^* &\in \text{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) \mid \langle \lambda^*, g(x_1^*, z_2) - g(x_1^*, x_2^*) \rangle \le 0, \quad z_2 \in X_2\}. \end{aligned}$$

We carry through some transformations of constraints for obtained problems. From the left-hand side inequality (70), we get $\langle \lambda - \lambda^*, g(x_1^*, x_2^*) + g(x_1^*, x_2^*) \rangle \geq 0 \ \forall \lambda \geq 0$. Assuming, at first $\lambda = 0$, and then $\lambda = 2\lambda^*$ in this inequality, we obtain $\langle \lambda^*, g(x_1^*, x_2^*) \rangle = 0$ and $\lambda^* \geq 0$. In view of said the last two problems get a kind

$$\begin{aligned} x_1^* &\in \text{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) \mid \langle \lambda^*, g(z_1, x_2^*) \rangle \leq 0, \ \ z_1 \in X_1\}, \\ x_2^* &\in \text{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) \mid \langle \lambda^*, g(x_1^*, z_2) \rangle \leq 0, \ \ z_2 \in X_2\}. \end{aligned} \tag{71}$$

The obtained statements mean that x_1^* and x_2^* are optimal points for functions $f_1(z_1, x_2^*) + \varphi_1(z_1)$ and $f_2(x_1^*, z_2) + \varphi_2(z_2)$ on sets $\{z_1 \mid \langle \lambda^*, g(z_1, x_2^*) \rangle \leq 0, \ \ z_1 \in X_1\}$ and $\{z_2 \mid \langle \lambda^*, g(x_1^*, z_2) \rangle \leq 0, \ \ z_2 \in X_2\}$ in equilibrium state. Uneasy to see that these points are optimal solutions of the same functions on sets of a kind $\{z_1 \mid \mid g(z_1, x_2^*) \leq 0, \ \ z_1 \in X_1\}$ and $\{z_2 \mid g(x_1^*, z_2) \leq 0, \ \ z_2 \in X_2\}$. Otherwise, under some regularity condition for these sets it is easily to receive inconsistency supposing that there exists another point x_1', for example for the first condition, such that $f_1(x_1', x_2^*) + \varphi_1(x_1') < f_1(x_1^*, x_2^*) + \varphi_1(x_1^*)$ and $g(x_1', x_2^*) \leq 0$. Multiplying the last inequality to non-negative vector λ^*, we get at once an inconsistency with above assertion. Thus, from (71) we have

$$\begin{aligned} x_1^* &\in \text{Argmin}\{f_1(z_1, x_2^*) + \varphi_1(z_1) \mid g(z_1, x_2^*) \leq 0, \ \ z_1 \in X_1\}, \\ x_2^* &\in \text{Argmin}\{f_2(x_1^*, z_2) + \varphi_2(z_2) \mid g(x_1^*, z_2) \leq 0, \ \ z_2 \in X_2\}. \end{aligned} \tag{72}$$

The equivalence of (65) and (67) is established.

6 Extragradient coupled constraints game methods

To solve the game (72) it is enough to solve the equilibrium problem (67). With this purpose we present the Lagrange function for (69) in new variables

$$\mathcal{L}(v, w, \lambda) = \Phi(v, w) + \varphi(w) + \langle \lambda, G(v, w) \rangle, \tag{73}$$

where $v \in \Omega$, $w \in \Omega$, $\lambda \geq 0$. The system of inequalities (70) in these variables looks like

$$\mathcal{L}(v^*, v^*, \lambda) \leq \mathcal{L}(v^*, v^*, \lambda^*) \leq \mathcal{L}(v^*, w, \lambda^*) \qquad \forall w \in \Omega, \quad \lambda \geq 0. \tag{74}$$

We rewrite this system of inequalities in the equivalent form

$$\begin{aligned} & v^* \in \text{Argmin}\{\Phi(v^*, w) + \varphi(w) + \langle \lambda^*, G(v^*, w) \rangle \mid w \in \Omega\}, \\ & \lambda_1^* \in \text{Argmax}\{\langle \lambda, G(v^*, v^*) \rangle \mid \lambda \geq 0\}. \end{aligned} \tag{75}$$

The obtained system of problems generates in turn necessary (sufficient in convex case) condition of a minimum both in the form of variational inequality and in the form of operator equation. Of course, both forms are equivalent, first of them looks like

$$
\begin{array}{rl}
& \langle \nabla_2\Phi(v^*, v^*) + \nabla\varphi(v^*) + \nabla_2 G^\top(v^*, v^*)\lambda^*, w - v^*\rangle \geq 0 \qquad \forall w \in \Omega, \\
- & \langle G(v^*, v^*), \lambda - \lambda^*\rangle \geq 0 \qquad \forall \lambda \geq 0,
\end{array} \tag{76}
$$

where $\nabla_2\Phi(v, w)$, $\varphi(w)$ are a partial gradient in w for any v and a gradient for functions $\Phi(v, w)$, $\varphi(w)$ respectively. $\nabla_2 G^\top(v, w)$ is $m_1 \times n$ matrix, where $\nabla_2 g_i(v, w)$, $i = 1, 2, \ldots, m$ are vector-lines.

The second form of necessary conditions for (74) takes the form of operator equation

$$
\begin{array}{rcl}
v^* & = & \pi_\Omega(v^* - \alpha(\nabla_2\Phi(v^*, v^*) + \nabla\varphi(v^*) + \nabla_2 G^\top(v^*, v^*)\lambda^*)), \\
\lambda^* & = & \pi_+(\lambda^* + (\alpha/2)G(v^*, v^*)),
\end{array} \tag{77}
$$

where $\pi_\Omega(\ldots)$, $\pi_+(\ldots)$ are projection operators of some vector into Ω and the positive orthant R^n_+ respectively, $\alpha > 0$ is a parameter like steplength.

Before to pass to discussing to solution methods of the system of equations (77) we consider properties of function $G(v, w)$ in more details . First of all we mark properties of symmetry for this function. Indeed, from $G(v, w) = g(z_1, x_2) + g(x_1, z_2) =$ $= g(x_1, z_2) + g(z_1, x_2) = G(w, v)$ it follows

$$
G(v, w) = G(w, v) \qquad \forall v \in \Omega, \;\; w \in \Omega. \tag{78}
$$

Differentiating identity (78) in w, we receive

$$
\nabla_2 G^\top(v, w) = \nabla_1 G^\top(w, v) \qquad \forall v \in \Omega, \;\; w \in \Omega, \tag{79}
$$

where $\nabla_1 G^\top(\cdot,\cdot)$, $\nabla_2 G^\top(\cdot,\cdot)$ are partial gradients (derivatives) in first and second variables.

We prove a key property of symmetrical function $G(v, w)$, namely: private in w gradient-restriction of function $G(v, w)$ on the diagonal of square $\Omega \times \Omega$ is equal to a half of gradient of restricted function $G(v, w)$ onto this diagonal

$$
2\nabla_2 G^\top(v, w)|_{v=w} = \nabla G^\top(v, v) \qquad \forall v \in \Omega. \tag{80}
$$

Indeed, by the definition of the differentiability for function $G(v, w)$ we get, Antipin 1998

$$
G(v + h, w + k) = G(v, w) + \nabla_1 G^\top(v, w)h + \nabla_2 G^\top(v, w)k + \omega(v, w, h, k), \tag{81}
$$

where $\omega(v, w, h, k)/(|h|^2 + |k|^2)^{1/2} \to 0$ as $|h|^2 + |k|^2 \to 0$. Let $w = v$ and $h = k$ be, then using (79), we get

$$
G(v + h, v + h) = G(v, v) + 2\nabla_2 G^\top(v, w)|_{v=w} h + \omega(v, h), \tag{82}
$$

where $\omega(v,h)/|h| \to 0$ as $|h| \to 0$. Since (82) is a particular case of (81) it means that gradient-restriction $2\nabla_2 G^\top(v,w)|_{v=w}$ is the gradient $\nabla G^\top(v,v)$ of $G(v,v)$, i.e. (80) is true.

If the function $G(v,w)$ is convex in w for any v, then the operator $\nabla_2 G^\top(v,w)$ is monotone in w for any v but the gradient-restriction $\nabla_2 G^\top(v,w)|_{v=w}$, generally speaking, is not monotone one. To make sure the monotonicity of gradient-restriction (is not mandatory for symmetrical functions) we introduce a class positive-semidefinite functions. To this end, we mark in

$$g_0(v,w) = \Phi(v,w), \qquad G(v,w) = (g_1(v,w), \dots, G_m(v,w))$$

and for each functions $g_i(v,w)$, $i = 0,1,\dots,m$, we extend (7), Antipin 1995.

A function $g_i(v,w)$ is called positive-semidefinite onto $\Omega \times \Omega$, if it obeys the inequality

$$g_i(w,w) - g_i(w,v) - g_i(v,w) + g_i(v,v) \geq 0 \qquad \forall w,v \in \Omega. \tag{83}$$

The condition of positive-semidefiniteness of (83) is sufficient to guarantee the monotonicity of gradient-restriction $\nabla_2 g_i(v,w)|_{v=w}$, if the function $g_i(v,w)$ is convex in w for any v. Indeed, using the system of inequalities (9) from (83) we get the monotonicity of gradient-restriction

$$\langle \nabla_2 g_i(w,w) - \nabla_2 g_i(v,v), w - v\rangle \geq 0 \qquad \forall v,w \in \Omega. \tag{84}$$

Using inequalities obtained, we transform separately third term in the first inequality (76). Taking into account the key property of symmetric functions (80) and convexity of vectorial function $G(v,v)$ componently, we have

$$\begin{aligned}\langle \nabla_2 G^\top(v^*,v^*)\lambda^*, w - v^*\rangle &= \frac{1}{2}\langle \lambda^*, \nabla G^\top(v^*,v^*)(w - v^*)\rangle \leq \\ &\leq \frac{1}{2}\langle \lambda^*, G(w,w) - G(v^*,v^*)\rangle \geq 0.\end{aligned}$$

In view of an obtained evaluation we rewrite the first inequality from (76) in the form

$$\langle \nabla_2 \Phi(v^*,v^*) + \nabla\varphi(v^*), w - v^*\rangle + (1/2)\langle \lambda^*, G(w,w) - G(v^*,v^*)\rangle \geq 0 \qquad \forall w \in \Omega. \tag{85}$$

If the operator $\nabla_2 \Phi(v,v) + \varphi(v)$ is monotone, then by virtue of (84) we get from (85)

$$\langle \nabla_2 \Phi(w,w) + \nabla\varphi(w), w - v^*\rangle + (1/2)\langle \lambda^*, G(w,w) - G(v^*,v^*)\rangle \geq 0 \qquad \forall w \in \Omega. \tag{86}$$

These estimates are underlying the convergence analysis of gradient-type methods to the equilibrium solutions, Antipin 2001 A.

To solve system of equations (77) we use the extragradient approach

$$\begin{aligned}\bar\lambda^n &= \pi_+(\lambda^n + (\alpha/2)G(v^n,v^n)), \\ \bar v^n &= \pi_\Omega(v^n - \alpha(\nabla_2 \Phi(v^n,v^n) + \nabla\varphi(v^n) + \nabla_2 G^\top(v^n,v^n)\bar\lambda^n)),\end{aligned} \tag{87}$$

$$\begin{array}{rcl}\lambda^{n+1} & = & \pi_+(\lambda^n + (\alpha/2)G(\bar{v}^n, \bar{v}^n)),\\ v^{n+1} & = & \pi_\Omega(v^n - \alpha(\nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n) + \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)\bar{\lambda}^n)),\end{array} \qquad (88)$$

The steplength α in (87), (88) is determined from the interval

$$0 < \varepsilon \leq \alpha < 1/\sqrt{2(L_1 + L_2 C)^2 + (1/2)L_3^2}, \qquad \varepsilon > 0, \qquad (89)$$

where constants L_1, L_2, L_3, C are determined in below.

For the justification of correctness of selecting out parameter α we receive evaluations of deviations for vectors $\bar{v}^n$ and v^{n+1}, $\bar{\lambda}^n$ and λ^{n+1} in (87), (88)

$$|\bar{\lambda}^n - \lambda^{n+1}| \leq (\alpha/2)|G(v^n, v^n) - G(\bar{v}^n, \bar{v}^n)| \leq (\alpha/2)L_3|v^n - \bar{v}^n|, \qquad (90)$$

$$\begin{array}{rl} & |\bar{v}^n - v^{n+1}| \leq \alpha(|\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) - \nabla_2\Phi(\bar{v}^n, \bar{v}^n) - \nabla\varphi(\bar{v}^n)| +\\ + & |\nabla_2 G^\top(v^n, v^n) - \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)||\bar{\lambda}^n|) \leq \alpha(L_1 + L_2|\bar{\lambda}^n|)|\bar{v}^n - v^n| \leq\\ \leq & \alpha(L_1 + L_2 C)|\bar{v}^n - v^n|,\end{array} \qquad (91)$$

where

$$\begin{array}{l}(|\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) - \nabla_2\Phi(\bar{v}^n, \bar{v}^n) - \nabla\varphi(\bar{v}^n)| \leq L_1|\bar{v}^n - v^n|,\\ |\nabla_2 G^\top(v^n, v^n) - \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)| \leq L_2|\bar{v}^n - v^n|, \qquad |\bar{\lambda}^n| \leq C \qquad \forall n \to \infty.\end{array}$$

We rewrite process (87), (88) in the form of variational inequalities.

$$\langle bar\lambda^n - \lambda^n - (\alpha/2)G(v^n, v^n), \lambda - \bar{\lambda}^n\rangle \geq 0 \qquad \forall \lambda \geq 0,$$

$$\langle \bar{v}^n - v^n + \alpha(\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) + \nabla_2 G^\top(v^n, v^n)\bar{\lambda}^n), v - \bar{v}^n\rangle \geq 0 \qquad \forall v \in \Omega, \quad (92)$$

and

$$v^n)), \lambda - \lambda^{n+1}\rangle \geq 0 \qquad \forall \lambda \geq 0,$$

$$\langle v^{n+1} - v^n + \alpha(\nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n) + \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)\bar{\lambda}^n), v - v^{n+1}\rangle \geq 0 \qquad \forall v \in \Omega, \quad (93)$$

We show that the process (87) – (89) converges monotonically under the norm to one of equilibrium solutions.

Theorem 3 *Suppose that a solution set of game problem* (72) *is non-empty, function* $\Phi(v, w)$, $G(v, w)$ *are positive-semidefinite and convex in* w *for any* v, $G(v, w)$ *is a symmetric function, the Lipschitz conditions hold in* (90), (91), *dual sequence* $|\bar{\lambda}^n| \leq$ $\leq C$ *is bounded for all* n, $\Omega \subseteq R^n$ *is convex closed set. Then, the sequence* v^n, λ^n, *generated by method* (87) – (89) *converges monotonically under the norm to one of the equilibrium solutions, i.e.* $v^n, \lambda^n \to v^*, \lambda^* \in \Omega^* \times R^N_+$ *as* $n \to \infty$.

Proof. By putting $w = v^*$ in (93), we get

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \alpha\langle\nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), v^* - v^{n+1}\rangle +\\ + & \alpha\langle\nabla_2 G^\top(\bar{v}^n, \bar{v}^n)\bar{\lambda}^n, v^* - v^{n+1}\rangle \geq 0.\end{array} \qquad (94)$$

Take $w = v^{n+1}$ in (92)

$$\langle \bar{v}^n - v^n + \alpha(\nabla_2\Phi(v^n, v^n) + \nabla\varphi(v^n) + \nabla_2 G^\top(v^n, v^n)\bar{\lambda}^n), v^{n+1} - \bar{v}^n\rangle \geq 0.$$

Hence

$$\begin{array}{rl} & \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \alpha\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), v^{n+1} - \bar{v}^n\rangle - \\ - & \alpha\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n) - \nabla_2\Phi(v^n, v^n) - \nabla\varphi(v^n), v^{n+1} - \bar{v}^n\rangle + \\ + & \alpha\langle \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)\bar{\lambda}^n, v^{n+1} - \bar{v}^n\rangle - \\ - & \alpha\langle (\nabla_2 G^\top(\bar{v}^n, \bar{v}^n) - \nabla_2 G^\top(v^n, v^n))\bar{\lambda}^n, v^{n+1} - \bar{v}^n\rangle \geq 0, \end{array}$$

or taking into account (91)

$$\begin{array}{rl} & \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \alpha\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), v^{n+1} - \bar{v}^n\rangle + \\ + & \alpha\langle \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)\bar{\lambda}^n, v^{n+1} - \bar{v}^n\rangle + \alpha^2(L_1 + L_2 C)^2|\bar{v}^n - v^n| \geq 0. \end{array} \quad (95)$$

We add inequalities (94) and (95)

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \\ + & \alpha\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), v^* - \bar{v}^n\rangle + \\ + & \alpha\langle \nabla_2 G^\top(\bar{v}^n, \bar{v}^n)\bar{\lambda}^n, v^* - \bar{v}^n\rangle + \\ + & \alpha^2(L_1 + L_2 C)^2|\bar{v}^n - v^n|^2 \geq 0. \end{array} \quad (96)$$

Using (80) and convexity of function $G(v, v)$ by virtue of (84), we transform apart the fourth term from (96)

$$\begin{array}{rl} & \langle \bar{\lambda}^n, \nabla_2 G(\bar{v}^n, \bar{v}^n)(v^* - \bar{v}^n)\rangle = (1/2)\langle \bar{\lambda}^n, \nabla G(\bar{v}^n, \bar{v}^n)(v^* - \bar{v}^n)\rangle \leq \\ \leq & (1/2)\langle \bar{\lambda}^n, G(v^*, v^*) - G(\bar{v}^n, \bar{v}^n)\rangle, \end{array}$$

then

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \\ + & \alpha\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), v^* - \bar{v}^n\rangle + \\ + & (\alpha/2)\langle \bar{\lambda}^n, G(v^*, v^*) - G(\bar{v}^n, \bar{v}^n)\rangle + \\ + & \alpha^2(L_1 + L_2 C)^2|\bar{v}^n - v^n|^2 \geq 0. \end{array}$$

We put $w = \bar{v}^n$ in inequality (86)

$$\langle \nabla_2\Phi(\bar{v}^n, \bar{v}^n) + \nabla\varphi(\bar{v}^n), \bar{v}^n - v^*\rangle + (1/2)\langle \lambda^*, G(\bar{v}^n, \bar{v}^n) - G(v^*, v^*)\rangle \geq 0.$$

Add two last inequalities

$$\begin{array}{rl} & \langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n\rangle + \\ + & (\alpha/2)\langle \bar{\lambda}^n - \lambda^*, G(v^*, v^*) - G(\bar{v}^n, \bar{v}^n)\rangle + \alpha^2(L_1 + L_2 C)^2|\bar{v}^n - v^n|^2 \geq 0. \end{array} \quad (97)$$

Consider first inequalities from (92) and (93). Put $\lambda = \lambda^*$ in (93)

$$\langle \lambda^{n+1} - \lambda^n, \lambda^* - \lambda^{n+1}\rangle - (\alpha/2)\langle G(\bar{v}^n, \bar{v}^n), \lambda^* - \lambda^{n+1}\rangle \geq 0 \quad (98)$$

и $\lambda = \lambda^{n+1}$ в (92)

$$\begin{aligned} &\langle \bar{\lambda}^n - \lambda^n, \lambda^{n+1} - \bar{\lambda}^n \rangle + (\alpha/2)\langle G(\bar{v}^n, \bar{v}^n) - G(v^n, v^n), \lambda^{n+1} - \bar{\lambda}^n \rangle - \\ - \ &(\alpha/2)\langle G(\bar{v}^n, \bar{v}^n), \lambda^{n+1} - \bar{\lambda}^n \rangle \geq 0, \end{aligned} \tag{99}$$

Second term in this inequality can be estimated by means of (90), and then we add both inequalities (98) and (99)

$$\begin{aligned} &\langle \lambda^{n+1} - \lambda^n, \lambda^* - \lambda^{n+1} \rangle + \langle \bar{\lambda}^n - \lambda^n, \lambda^{n+1} - \bar{\lambda}^n \rangle + \\ + \ &(\alpha/2)^2 L_3^2 |\bar{v}^n - v^n|^2 - (\alpha/2)\langle G(\bar{v}^n, \bar{v}^n), \lambda^* - \bar{\lambda}^n \rangle \geq 0. \end{aligned}$$

Using the relations $\langle \bar{\lambda}^n, G(v^*, v^*) \rangle \leq 0$, $\langle \lambda^*, G(v^*, v^*) \rangle = 0$, we rewrite the latter inequality in the form

$$\begin{aligned} &\langle \lambda^{n+1} - \lambda^n, \lambda^* - \lambda^{n+1} \rangle + \langle \bar{\lambda}^n - \lambda^n, \lambda^{n+1} - \bar{\lambda}^n \rangle + \\ + \ &\left(\frac{\alpha}{2}\right)^2 L_3^2 |\bar{v}^n - v^n|^2 + \frac{\alpha}{2}\langle G(v^*, v^*) - G(\bar{v}^n, \bar{v}^n), \lambda^* - \bar{\lambda}^n \rangle \geq 0, \end{aligned} \tag{100}$$

We add inequalities (97) and (100)

$$\begin{aligned} &\langle v^{n+1} - v^n, v^* - v^{n+1} \rangle + \langle \bar{v}^n - v^n, v^{n+1} - \bar{v}^n \rangle + \\ + \ &\langle \lambda^{n+1} - \lambda^n, \lambda^* - \lambda^{n+1} \rangle + \langle \bar{\lambda}^n - \lambda^n, \lambda^{n+1} - \bar{\lambda}^n \rangle + \\ + \ &\alpha^2 (L_1 + L_2 C)^2 |\bar{v}^n - v^n|^2 + (\alpha/2)^2 L_3^2 |\bar{v}^n - v^n|^2 \geq 0. \end{aligned}$$

By means of identity (33) we expand first four scalar products into the sum of squares

$$\begin{aligned} \frac{1}{2}|v^{n+1} - v^*|^2 \ &+ \ \frac{1}{2}|\lambda^{n+1} - \lambda^*|^2 + \frac{1}{2}|v^{n+1} - \bar{v}^n|^2 + \frac{1}{2}|\bar{v}^n - v^n|^2 + \\ &+ \ \frac{1}{2}|\lambda^{n+1} - \bar{\lambda}^n|^2 + \frac{1}{2}|\bar{\lambda}^n - \lambda^n|^2 + \\ &+ \ \alpha^2 \left((L_1 + L_2 C)^2 + \left(\frac{1}{2}\right)^2 L_3^2 \right) |\bar{v}^n - v^n|^2 \leq \\ &\leq \ \frac{1}{2}|v^n - v^*|^2 + \frac{1}{2}|\lambda^n - \lambda^*|^2. \end{aligned} \tag{101}$$

From here

$$\begin{aligned} &|v^{n+1} - v^*|^2 + |\lambda^{n+1} - \lambda^*|^2 + |v^{n+1} - v^n|^2 + d|\bar{v}^n - v^n|^2 + \\ + \ &|\lambda^{n+1} - \lambda^n|^2 + |\bar{\lambda}^n - \lambda^n|^2 \leq |v^n - v^*|^2 + |\lambda^n - \lambda^*|^2, \end{aligned} \tag{102}$$

where $d = 1 - 2\alpha^2((L_1 + L_2 C)^2 + (1/2)^2 L_3^2) > 0$, by virtue of (89).

Summing (102) from $n = 0$ up to $n = N$, we obtain

$$\begin{aligned} &|v^{N+1} - v^*|^2 + |\lambda^{N+1} - \lambda^*|^2 + \sum_{k=0}^{k=N} |v^{k+1} - \bar{v}^k|^2 + \\ + \ &d \sum_{k=0}^{k=N} |\bar{v}^k - v^k|^2 + \sum_{k=0}^{k=N} |\lambda^{k+1} - \lambda^k|^2 \sum_{k=0}^{k=N} |\bar{\lambda}^k - \lambda^k|^2 \leq |v^0 - v^*|^2 + |\lambda^0 - \lambda^*|^2. \end{aligned}$$

From obtained inequality it follows the boundedness of trajectory

$$|v^{N+1}-v^*|^2+\frac{1}{2}|\lambda^{N+1}-\lambda^*|^2\le|v^0-v^*|^2+\frac{1}{2}|\lambda^0-\lambda^*|^2,$$

and the convergence of series

$$\begin{array}{ll}\sum_{k=0}^{\infty}|v^{k+1}-\bar v^k|^2<\infty, & \sum_{k=0}^{\infty}|\bar v^k-v^k|^2<\infty,\\ \sum_{k=0}^{\infty}|\lambda^{k+1}-\bar\lambda^k|^2<\infty, & \sum_{k=0}^{\infty}|\bar\lambda^k-\lambda^k|^2<\infty,\end{array}$$

and, consequently, tend to zero of quantities

$$|v^{n+1}-\bar v^n|^2\to 0,\quad |\bar v^n-v^n|^2\to 0,\quad |\lambda^{n+1}-\bar\lambda^n|^2\to 0,\quad |\bar\lambda^n-\lambda^n|^2\to 0,\quad n\to\infty.$$

Since the sequence v^n,λ^n is bounded, there exist a point v',λ' such that $v^{n_i}\to v'$, $\lambda^{n_i}\to\lambda'$ при $n_i\to\infty$.

We consider inequalities (92), (93) for all $n_i\to\infty$ and, passing to a limit we get

$$\begin{array}{l}\langle\nabla_2\Phi(v',v')+\nabla\varphi(v')+\nabla_2 G^{\top}(v',v')\lambda',w-v'\rangle\ge 0\quad\forall w\in\Omega,\\ \langle -G(v',v'),\lambda-\lambda'\rangle\ge 0\quad\forall\lambda\ge 0.\end{array}$$

The inequalities obtained coincide to (76), then $v'=v^*\in\Omega^*$, $\lambda'=\lambda^*\ge 0$, i.e., any limit point of v^n,λ^n is an equilibrium solution to the problem. The monotonicity condition of decreasing value $|v^n-v^*|+|\lambda^n-\lambda^*|$ provides of uniqueness of limit point, i.e. the convergence of $v^n\to v^*$, $\lambda^n\to\lambda^*$ as $n\to\infty$. The theorem is proved.

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