Computational Mathematics and Mathematical Physics. Vol. 37. No. 11, 1997, pp. 1285–1296.
Translated from Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki.
Vol. 37. No. 11, 1997, pp. 1327–1339.

Equilibrium Programming: Proximal Methods

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Revised October 10, 2004

Abstract — The equilibrium-programming problem is formulated. Its relation to game settings is discussed. To solve this problem, implicit and explicit proximal-regularization methods using conventional and modified Lagrange functions are suggested. The convergence of these methods to the equilibrium solutions is proven.

1. FORMULATION OF THE PROBLEM

The equilibrium-programming problem can be formulated as follows. Find a fixed point $v^* \in \Omega^*$ that satisfies the following extremal inclusion with functional constraints:

$$v^* \in \text{Argmin}\{\Phi(v^*, w) \mid g(w) \leq 0, \ \ w \in \Omega\}. \tag{1.1}$$

Here, the function $\Phi(v, w)$ is defined on the product space $\mathbb{R}^n \times \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$ is a convex closed set. We assume that $\Phi(v, w)$ is convex with respect to the variable $w \in \Omega$ for every $v \in \Omega$. The vector-valued function $g(w)$ has the dimension m. Every component of this function is convex. The variable $v \in \Omega$ in (1.1) plays the role of a parameter, and $w \in \Omega$ is the optimization variable. We also assume that the extremal (marginal) mapping $w(v) \equiv \text{argmin}\{\Phi(v, w) \mid g(w) \leq 0, \ w \in \Omega\}$ is defined for all $v \in \Omega$, and the set of solutions $\Omega^* \subset \Omega$ to the original problem is nonempty. By virtue of the Kakutani theorem, the latter assumption is always fulfilled if Ω is a convex compact set and $\Phi(v, w)$ is lower semicontinuous with respect to v and convex with respect to w (see [1]). In this case, the extremal mapping is upper semicontinuous and translates every point from Ω into a closed convex set.

Problem (1.1) can be considered as a convolution, which includes many game settings. Indeed, assume that the inequalities

$$L(x^*, y) \leq L(x^*, p^*) \leq L(z, p^*) \qquad \forall z \in Q \subseteq \mathbb{R}^n \quad \forall y \in P \subseteq \mathbb{R}^m \tag{1.2}$$

define a saddle point x^*, p^*, where $L(x, p)$ is a convex-concave function, $Q = \{z \mid g_1(z) \leq$ $\leq 0, \ z \in Q_1\}$, $P = \{y \mid g_2(y) \leq 0, \ y \in P_1\}$, and $g_1(z)$ and $g_2(y)$ are convex vector-valued functions. We use the notation $w = (z, y)$, $v = (x, p)$, $G(w) = (g_1(z), g_2(y))$, and the normalized function $\Phi(v, w) = L(z, p) - L(x, y)$. This notation allows one to represent (1.2) as (1.1) (see [2]). The solution of the saddle system (1.2) with the normalized function $\Phi(v, w) = L(z, p) - L(x, y))$ is equivalent to the solution of (1.1).

A more general situation of an n-person game with the Nash equilibrium can also be reduced to (1.1). Indeed, let $f_i(x_i, x_{-i})$ be the payoff function of the ith player, $i \in I$. This function depends both on the strategy of this player $x_i \in X_i$, where $X_i = (x_i)_{i \in I}$, and on the strategies of all other players $x_{-i} = (x_j)_{j \in I \setminus i}$. An equilibrium of the n-person game is a solution to the extremal inclusions

$$x_i^* \in \text{Argmin}\{f_i(x_i, x_{-i}^*) \mid x_i \in X_i\}. \tag{1.3}$$

Consider a normalized function

$$\Phi(v, w) = \sum_{i=1}^{n} f_i(x_i, x_{-i}),$$

where $v = (x_{-i})$, $w = (x_i)$, $g(w) = (g_i(x_i))$, $i = 1, 2, \ldots, n$, and $\Omega = X_1 \times X_2 \times \ldots \times X_n$; here $(v, w) = (x_i, x_{-i}) \in \Omega \times \Omega$. Using this function, we can write (1.3) as (1.1).

Many inverse optimization problems [3] can also be represented as (1.1). Indeed, consider the inverse problem of convex programming

$$x^* \in \operatorname{Argmin}\{\langle \lambda^*, f(x)\rangle \mid g(x) \le 0, \ \ x \in Q\}, \qquad G(x^*) \le d. \tag{1.4}$$

In this problem, one must choose nonnegative coefficients of the linear convolution $\lambda = = \lambda^*$ so that the optimal solution $x = x^*$ corresponding to these weights belongs to the preassigned convex set. In particular, this set may contain only one point. It is assumed that all functions in this problem are convex.

System (1.4) can be represented as a two-person game with the Nash equilibrium:

$$\begin{aligned} x^* &\in \operatorname{Argmin}\{\langle \lambda^*, f(x)\rangle \mid g(x) \le 0, \ \ x \in Q\}, \\ p^* &\in \operatorname{Argmin}\{\langle p, G(x^*) - d\rangle \mid p \ge 0\}. \end{aligned} \tag{1.5}$$

Indeed, from (1.5), we deduce

$$\langle p^* - p, G(x^*) - d\rangle \le 0, \qquad p \ge 0. \tag{1.6}$$

First, we set $p = 0$ in (1.6), then we set $p = 2p^*$, which results in $\langle p^*, G(x^*) - d\rangle = 0$. Now, if we suppose that some component of the vector $G(x^*) - d$ on the right-hand side of the inequality

$$0 = \langle p^*, G(x^*) - d\rangle \le \langle p, G(x^*) - d\rangle \tag{1.7}$$

is negative, then, since $p \le 0$ is arbitrary, it is easy to obtain a contradiction with (1.7). Hence, $G(x^*) - d \le 0$. Thus, a solution to (1.5) is a solution to (1.4). The converse statement is also true. Problem (1.5), in turn, can be reduced to (1.1) with the help of the normalized function. Hence, the original inverse optimization problem (1.4) can be represented as a problem of calculating a fixed point of the extremal mapping (1.1).

2. SKEW-SYMMETRIC FUNCTIONS

Formula (1.1) implies that any fixed point satisfies the inequality

$$\Phi(v^*, v^*) \le \Phi(v^*, w) \qquad \forall w \in D, \tag{2.1}$$

where $D = \{w \mid g(w) \le 0, \ \ w \in \Omega\}$ is an admissible set. This inequality is equivalent to the definition of a fixed point. Since $\Phi^* = \inf\{\Phi(w, w) \mid w \in D\} \le \Phi(v^*, v^*)$, formula (2.1) immediately implies the inequality of Ky Fan [4]

$$\inf\{\Phi(w, w) \mid w \in D\} \le \Phi(v^*, w). \tag{2.2}$$

This inequality is equivalent to the Kakutani theorem [1] and describes the existence of a fixed point of (1.1).

In [5, 6], an inequality that describes a saddle property of a fixed point was suggested that allows one to approximate the equilibrium solution to (1.1) with any degree of accuracy by continuous or iterative processes. This inequality has the form

$$\Phi(w, v^*) \leq \Phi(w, w) \qquad \forall w \in D. \tag{2.3}$$

If we rewrite (2.3) in a more general form

$$\Phi(w, v^*) \leq \sup\{\Phi(w, w) \mid w \in D\}$$

and compare it with the Ky Fan inequality (2.2), then we can see that these two inequalities are related by a mutual symmetry. When $\sup\{\dots\} = \inf\{\dots\} = \Phi(v^*, v^*)$, these two inequalities form a saddle system and, therefore, can be considered as a generalized saddle system.

Because of the nonconstructive character of inequality (2.3) (it contains the unknown vector v^*), we introduce a class of functions for which condition (2.3) is always fulfilled.

Definition 1. *A function $\Phi(v, w)$ from $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^1$ is called skew-symmetric on $\Theta \times \Theta$ if it satisfies the inequality*

$$\Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \geq 0 \tag{2.4}$$

for all $w \in \Theta$ and all $v \in \Theta$. If the inequality

$$\Phi(w, w) - \Phi(w, v^*) - \Phi(v^*, w) + \Phi(v^*, v^*) \geq 0 \tag{2.5}$$

is valid for all w from a neighborhood of the solution $v^* \in \Omega^*$, then the function $\Phi(v, w)$ is called skew-symmetric with respect to the equilibrium point.

Further, we assume that the set Θ coincides with either Ω or D. The class of skew-symmetric functions thus defined is nonempty. It is easy to verify (see [2]) that the normalized function $\Phi(v, w) = L(z, p) - L(x, y)$, $w = (z, y)$, $v = (x, p)$ of the saddle problem (1.2) is skew-symmetric.

For skew-symmetric functions, (2.3) is always fulfilled. Indeed, setting $v = v^* \in \Omega^*$ in (2.4) and taking into account (2.1), we obtain (2.3).

Skew-symmetric functions have properties that can be considered as analogues of the monotonicity of the gradient and the nonnegativity of the second derivative for convex functions.

Property 1. *If a function $\Phi(v, w)$ is skew-symmetric and convex with respect to its second argument, then its partial gradient $\nabla\Phi_w(v, v)$ is monotonic on the diagonal of the square $\Theta \times \Theta$:*

$$\langle \nabla\Phi_w(w, w) - \nabla\Phi_w(v, v), w - v \rangle \geq 0 \qquad \forall w \in \Theta, \quad v \in \Theta.$$

This inequality follows from (2.4) by virtue of the convexity of $\Phi(v, w)$ with respect to w:

$$\langle \nabla f(x), y - x \rangle \leq f(y) - f(x) \leq \langle \nabla f(y), y - x \rangle \tag{2.6}$$

for all x and y from a certain set.

If we set $v = v^*$ (2.6) and use (2.1) in the form

$$\langle \nabla\Phi_w(v^*, v^*), w - v^*\rangle \geq 0 \qquad \forall w \in \Theta,$$

then we obtain the following inequality:

$$\langle \nabla\Phi_w(w, w), w - v^*\rangle \geq 0 \qquad \forall w \in \Theta. \tag{2.7}$$

Inequality (2.7) is equivalent to (2.3) if the function $\Phi(v, w)$ is differentiate and convex with respect to w. For convex optimization problems, there is an analogue of this inequality. It can be written as $\langle \nabla\Phi(w), w - v^*\rangle \geq 0 \ \ \forall w \in \Theta$ (see [7]).

Property 2. *The mixed derivative $\nabla^2\Phi_{wv}(v, v)$ of the skew-symmetric function $\Phi(v, w)$ is nonnegative on the diagonal of the square $\Theta \times \Theta$:*

$$\langle \nabla^2\Phi_{wv}(v, v)h, h\rangle \geq 0 \qquad \forall h \in \mathbb{R}^n. \tag{2.8}$$

Condition (2.3) imposes a certain constraint on the behavior of the objective function in a neighborhood of the equilibrium. However, some problems (see the examples below) satisfy conditions that are even more restrictive than (2.3). Let v^* be an isolated equilibrium; then, for certain classes of problems, the following inequality holds:

$$\Phi(w, w) - \Phi(w, v^*) \geq \gamma|w - v^*|^{1+\nu} \qquad \forall w \in \Theta. \tag{2.9}$$

Here, $\nu \in [0, \infty)$ is a parameter, and $\gamma > 0$ is a constant. If $\nu = 0$, then we have a sharp equilibrium; if $\nu = 1$, then the equilibrium is quadratic.

3. EXAMPLES

In this section, we present examples that illustrate the diversity of problems for which condition (2.3) is fulfilled.

1. Quadratic equilibrium. Consider the problem of finding a fixed point of the quadratic extremal inclusion

$$v^* \in \text{Argmin}\left\{\frac{1}{2}\langle Nw, w\rangle + \langle Mv^* + m, w\rangle \mid w \in \Omega\right\}, \tag{3.1}$$

where N and M are nonnegative matrices; i.e., $\langle Nv, v\rangle \geq 0$ and $\langle Mv, v\rangle \geq 0$ for all $v \in \mathbb{R}^n$. We also assume that N is a symmetrical matrix. Consider

$$\begin{aligned}
& \Phi(w, w) - \Phi(w, v) - \Phi(v, w) + \Phi(v, v) \\
= \ & \frac{1}{2}\langle Nw, w\rangle + \langle Mw, w\rangle + \langle m, w\rangle - \frac{1}{2}\langle Nv, v\rangle - \langle Mv, v\rangle - \langle m, v\rangle - \\
- \ & \frac{1}{2}\langle Nw, w\rangle - \langle Mv, w\rangle - \langle m, w\rangle + \frac{1}{2}\langle Nv, v\rangle + \langle Mv, v\rangle + \langle m, v\rangle = \\
= \ & \langle M(w - v), w - v\rangle \geq 0 \quad \forall w \in \Omega \quad \forall v \in \Omega.
\end{aligned}$$

This implies that, if the matrix M is nonnegative, then $\Phi(w, v)$ in (3.1) satisfies (2.4). If M is strongly positive, then (2.9) is valid with $\nu = 1$.

2. The Cournot diopoly. This basic example is a model of the behavior of two monopolists that produce the same goods and compete in the same market. Simplifying the situation (for details, see [8]), we consider the diopoly as a game of two persons whose loss functions are defined by the formulas

$$f_1(z, y) = z(z + y - u), \quad f_2(z, y) = y(z + y - u), \tag{3.2}$$

where $z \in [0, u]$, $y \in [0, u]$, $u > 0$, and z and y are the amounts of goods produced by the first and second participant, respectively. If the second participant produces y^* units of goods, then the first participant brings $z^* = (u - y^*)/2$ units of goods to the market, which minimizes his functional expenses: $f_1(z, y^*) = = z(z + y^* - u)$. The second participant pursues the same strategy $y^* = (u - z^*)/2$ if he is informed that the first player has brought z^* units of goods to the market. The fixed point of equilibrium of the diopoly is the pair $z^* = u/3$, $y^* = u/3$. The expenses of the players are equal to $-u^2/9$. The Cournot diopoly is a quadratic game and, consequently, can be represented as (3.1). The matrix M does not satisfy the condition $\langle Mh, h\rangle \geq 0$ for all $h \in \mathbb{R}^n$. Nevertheless, this problem satisfies condition (2.3). Let its verify this. We write the normalized function $\Phi(v, w)$ for problem (3.2). For this purpose, we denote the variable y in the first formula (3.2) by p. We can do so, because we minimize $f_1(z, y)$ with respect to z. For the same reason, we denote the variable z in the formula for $f_2(z, y)$ by x. We introduce new variables $w = (z, y)$ and $v = (x, p)$, and write the normalized function for the game (3.2):

$$\Phi(v, w) = (z, y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} + (x, p)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} - (u, u)\begin{pmatrix} z \\ y \end{pmatrix}.$$

Let $w = v^* = (x^*, p^*)$ and $v = w = (z, y)$; then, (2.3) can be written as

$$\begin{aligned} &(x^*, p^*)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x^* \\ p^* \end{pmatrix} + (z, y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x^* \\ p^* \end{pmatrix} - (u, u)\begin{pmatrix} x^* \\ p^* \end{pmatrix} \leq \\ \leq\quad &(z, y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} + (z, y)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z \\ y \end{pmatrix} - (u, u)\begin{pmatrix} z \\ y \end{pmatrix}, \end{aligned}$$

This implies the inequality $(x^*)^2 + (p^*)^2 + zp^* + yx^* - ux^* - up^* \leq z^2 + y^2 + 2zy - uz - uy$. Since $x^* = u/3$ and $p^* = u/3$, we have $-4u^2/9 + u(z+y)/3 \leq (z+y)^2 - u(z+y)$. Finally, $0 \leq [(z + y) - -2u/3]^2$. Hence, the Cournot Diopoly game satisfies condition (2.3).

3. The prisoner's dilemma. Consider a finite game in which both participants have their sets of strategies, each of which consists of two elements: $\{I, II\}$ and $\{1, 2\}$. Assume that both participants are imprisoned for the crimes committed, and each of them has two possibilities: sincere confession (strategies II and 2) and a denial (strategies I and 1). Their payoff functions take the following values: $0, a, b, c$, where $0 < a < b < c$. Each of these numbers is treated as "a years of imprisonment". The matrix of the game is

	1	2
I	a, a	$c, 0$
II	$0, c$	b, b

This matrix implies that, if both participants either simultaneously deny their crimes or confess, then they are imprisoned for a or b years each. If one of them confesses, while the other denies his crime, then the first participant is released, and the other is imprisoned for c years.

The pair of strategies $(II, 2)$ is a (noncooperative) equilibrium state of this game. Indeed, let us verify (2.1):

$$f_1(II, 2) = b < f_1(I, 2) = c, \qquad f_2(II, 2) = b < f_2(II, 1) = c.$$

A large number of scientists have tested their ideas and methods by applying them to this game. Let us also test our method. Let us verify that this problem satisfies condition (2.3). Let $\Phi(v, w) = f_1(z, p) + f_2(x, y)$, where $v = (x, p)$ and $w = (z, y)$; then, (2.3) can be rewritten as

$$f_1(x^*, y) + f_2(z, p^*) \leq f_1(z, y) + f_2(z, y),$$

where $x^*, p^* = (II, 2)$. Hence $f_1(II, y) + f_2(z, 2) \leq f_1(z, y) + f_2(z, y)$. Here, the variable z ranges over the set $\{I, II\}$, and the variable y ranges over [1, 2]. Calculating (2.3) at the points $z = I,\ y = 1$; $z = II,\ y = 1$; $z = I,\ y = 2$; and $z = II,\ y = 2$, successively, we obtain the following relations: $0 \leq 2a$; $b = b$; $b \leq c$; $2b = 2b$. Since these relations are true, we have proven that the Prisoner's Dilemma game satisfies condition (2.3).

4. PROXIMAL-REGULARIZATION METHODS

Before discussing the methods for solving (1.1), we will make the following remark. Let us write (2.1) and (2.3) as

$$\Phi(w, v^*) - \Phi(w, w) \leq \Phi(v^*, v^*) - \Phi(v^*, v^*) \leq \Phi(v^*, w) - \Phi(v^*, v^*) \qquad \forall w \in D. \quad (4.1)$$

Consider the function $\Psi(v, w) = \Phi(v, w) - \Phi(v, v)$ and, with its help, write (4.1) as

$$\Psi(w, v^*) \leq \Psi(v^*, v^*) \leq \Psi(v^*, w) \ \ \forall w \in D.$$

The last system of inequalities implies that v^*, v^* is a saddle point for the function $\Psi(v, w)$. This circumstance is, however, of little use, because the function $\Psi(v, w)$ cannot be applied for calculating saddle points, since the saddle methods [2] require convexity with respect to one variable and concavity with respect to another. Although the function $\Psi(v, w)$ is convex with respect to w, it is not concave with respect to v. If we still wish to develop methods for calculating a saddle for $\Psi(v, w)$, then these methods must contain the shift procedure with respect to both v and w; this circumstance implies that the corresponding method has poor convergence properties, because the function is not concave with respect to the variable v. In our method, an equilibrium (in particular, a saddle) is approached by recalculating the iterations with respect to one variable w alone, and the process converges monotonically to the equilibrium (in the norm of the space) in the general case.

Following the usual scheme of convex programming, we introduce the Lagrange function for problem (1.1):

$$L(v^*, w, p) = \Phi(v^*, w) + \langle p, g(w) \rangle, \qquad w \in \Omega, \quad p \geq 0.$$

If the functional constraints are regular (e.g., if the Slater conditions are fulfilled), this problem can be transformed into the problem of calculating a saddle point of the Lagrange function $L(v^*, w, p)$, i.e.,

$$\begin{aligned} \Phi(v^*, v^*) + \langle p, g(v^*) \rangle \ &\leq \ \Phi(v^*, v^*) + \langle p^*, g(v^*) \rangle \leq \\ &\leq \ \Phi(v^*, w) + \langle p^*, g(w) \rangle \qquad \forall w \in \Omega, \quad p \geq 0. \end{aligned} \quad (4.2)$$

Let us write this system in the following equivalent form:

$$\begin{aligned} &v^* \in \mathrm{Argmin}\{\Phi(v^*, w) + \langle p^*, g(w)\rangle \mid w \in \Omega\}, \\ &p^* = \pi_+(p^* + \alpha g(v^*)), \end{aligned} \tag{4.3}$$

where $\pi_+(\ldots)$ is the operator that projects a vector to the positive orthant $\mathbb{R}^n_+$.

We consider two sorts of methods for solving the equilibrium problem (4.3): explicit and implicit ones. Implicit methods are iteration processes such that, at every iteration, an auxiliary regularized equilibrium problem is solved. In this approach, the original problem, which is, as a rule, degenerate, is replaced by a sequence of regularized equilibrium problems. To solve the latter problems, we use the methods developed in [5, 6] for calculating the fixed points of extremal mappings. Under some constraints on the data of the problem, we prove that the sequence of regularized equilibrium solutions converges to the solution to the original problem.

Explicit methods use more complicated iteration formulas, but, at every iteration, one has to solve a comparatively simple problem of optimizing a strongly convex function on a simple set. It is possible to prove that these methods converge if the length of the step is bounded by a constant.

The proximal-regularization method, based on a modified Lagrange function, is one of the methods for solving the equilibrium problem (1.1) Its iteration formulas are

$$v^{n+1} = \mathrm{argmin}\left\{\frac{1}{2}|w - v^n|^2 + \alpha M(v^{n+1}, w, p^n) \mid w \in \Omega\right\}, \tag{4.4}$$

$$p^{n+1} = \pi_+(p^n + \alpha g(v^{n+1})), \tag{4.5}$$

where

$$M(v, w, p) = \Phi(v, w) + \frac{1}{2\alpha}|\pi_+(p + \alpha g(w))|^2 - \frac{1}{2\alpha}|p|^2$$

is defined for all $v, w \in \mathbb{R}^n \times \mathbb{R}^n$ and $p \geq 0$. Here, v^n, p^n is the approximation, and v^{n+1}, p^{n+1} is the desired solution. Equation (4.4) contains the unknowns v^{n+1} on both its left- and right-hand sides (an implicit scheme). Using the notation $R(v, w, v^n, p^n) = = |w - v^n|^2/2 + \alpha M(v, w, p^n)$, we can write (4.4) as a problem of calculating a fixed point of the extremal mapping

$$v^{n+1} = \mathrm{argmin}\{R(v^{n+1}, w, v^n, p^n) \mid w \in \Omega\}.$$

The methods for solving this problem were investigated in [5, 6].

It is natural to consider (4.4), (4.5) as a proximal-regularization method in which the (degenerate) function $\Phi(v, w)$ is regularized with the help of the quadratic term $|w-v^n|^2/2$, and the functional constraints of the problem are taken into account with the help of the modified Lagrange function.

Let us discuss the problems of the convergence of this method. Recall that the functions $\Phi(v, w)$ and $g(w)$ are assumed to be convex with respect to the variable w, but not necessarily differentiable. In the latter case, these functions have subdifferentials in their domains. It is well known that, at a minimum point, the subdifferential of a function contains a subgradient such that the linear function corresponding to this subgradient is nonnegative on an admissible convex set. Adapting this situation for (4.4) and (4.5), where v^{n+1} and p^{n+1} are the minimum points of the corresponding functions, we represent the process as variational inequalities

$$\langle v^{n+1} - v^n + \alpha\nabla\Phi_w(v^{n+1}, v^{n+1}) + \alpha\nabla g^\top(v^{n+1})\pi_+(p^n + \alpha g(v^{n+1})), w - v^{n+1}\rangle \geq 0, \tag{4.6}$$

$$\langle p^{n+1} - p^n - \alpha g(v^{n+1}), p - p^{n+1}\rangle \geq 0. \tag{4.7}$$

These inequalities are valid for all $w \in \Omega$ and all $p \geq 0$. Here, $\nabla\Phi_w(v,w)$ is the vector-subgradient of the function $\Phi(v,w)$ with respect to the variable w, and $\nabla g^\top(v)$ is the transposed matrix in which every column is the vector-subgradient of the corresponding scalar function of the vector $g(v)$.

Let us show that method (4.4), (4.5) converges monotonically in the norm to an equilibrium solution to the problem, assuming that the set Θ in (2.5) is identical with Ω.

Theorem 1. *If the set of solutions to* (1.1) *is nonempty and satisfies condition* (2.5) *for all $w \in \Omega$, the objective function $\Phi(v,w)$ is continuous with respect to v and convex with respect to w for every $v \in \Omega$, $g(w)$ is a convex vector-valued function, and $\Omega \subseteq \mathbb{R}^n$ is a convex closed set, then the sequence v^n generated by* (4.4), (4.5) *converges to cm equilibrium solution monotonically in the norm, i.e., $v^n \to v^* \in \Omega^*$ as $n \to \infty$.*

Proof. Setting $w = v^*$ in (4.6) and taking into account (4.5), we obtain

$$\langle v^{n+1} - v^n + \alpha\nabla\Phi_w(v^{n+1}, v^{n+1}) + \alpha\nabla g^\top(v^{n+1})p^{n+1}, v^* - v^{n+1}\rangle \geq 0. \tag{4.8}$$

Using the convexity inequalities (2.6), we transform (4.8) as follows:

$$\begin{aligned}\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle &+ \alpha[\Phi(v^{n+1}, v^*) - \Phi(v^{n+1}, v^{n+1})]+\\ &+ \alpha\langle p^{n+1}, g(v^*) - g(v^{n+1})\rangle \geq 0.\end{aligned} \tag{4.9}$$

Setting $w = v^{n+1}$ in (4.2), we write the last inequality as

$$\Phi(v^*, v^{n+1}) - \Phi(v^*, v^*) + \langle p^*, g(v^{n+1}) - g(v^*)\rangle \geq 0. \tag{4.10}$$

Summing (4.9) and (4.10), we infer that

$$\begin{aligned}\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle &- \alpha[\Phi(v^{n+1}, v^{n+1}) - \Phi(v^{n+1}, v^*)] - \Phi(v^*, v^{n+1})+\\ &+ \Phi(v^*, v^*)) + \alpha\langle p^{n+1} - p^*, g(v^*) - g(v^{n+1})\rangle \geq 0.\end{aligned} \tag{4.11}$$

Further, we put $p = p^*$ in (4.7):

$$\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle - \alpha\langle g(v^{n+1}), p^* - p^{n+1}\rangle \geq 0. \tag{4.12}$$

Let us sum (4.11) and (4.12). Taking into account the formulas $\langle p^{n+1}, g(v^*)\rangle \leq 0$, $\langle p^*, g(v^*)\rangle = 0$, and (2.5), we obtain

$$\langle v^{n+1} - v^n, v^* - v^{n+1}\rangle + \langle p^{n+1} - p^n, p^* - p^{n+1}\rangle \geq 0.$$

Using the identity

$$|x_1 - x_3|^2 = |x_1 - x_2|^2 + 2\langle x_1 - x_2, x_2 - x_3\rangle + |x_2 - x_3|^2, \tag{4.13}$$

we expand the scalar products on the left-hand side of the obtained inequality in a sum of squares:

$$\begin{aligned}|v^{n+1} - v^*|^2 &+ |p^{n+1} - p^*|^2 + |v^{n+1} - v^n|^2 + |p^{n+1} - p^n|^2 \leq\\ &\leq |v^n - v^*|^2 + |p^n - p^*|^2.\end{aligned} \tag{4.14}$$

Summing inequality (4.14) with respect to n from 0 to N, we obtain the inequality

$$\begin{aligned}|v^{N+1}-v^*|^2 \quad + \quad & |p^{N+1}-p^*|^2+\sum_{k=0}^{k=N}|v^{k+1}-v^k|^2+\sum_{k=0}^{k=N}|p^{k+1}-p^k|^2\leq \\ \leq \quad & |v^0-v^*|^2+|p^0-p^*|^2.\end{aligned}$$

which implies that the trajectory is bounded, i.e.,

$$|v^{N+1}-v^*|^2+|p^{N+1}-p^*|^2\leq |v^0-v^*|^2+|p^0-p^*|^2, \tag{4.15}$$

and the series

$$\sum_{k=0}^{\infty}|v^{k+1}-v^k|^2<\infty, \qquad \sum_{k=0}^{\infty}|p^{k+1}-p^k|^2<\infty,$$

converge; hence, the following quantities tend to zero:

$$|v^{n+1}-v^n|^2\to 0, \qquad |p^{n+1}-p^n|^2\to 0, \qquad n\to\infty. \tag{4.16}$$

Since the sequence v^n, p^n is bounded, there exists an element v', p' such that $v^{n_i}\to v'$ and $p^{n_i}\to p'$ as $n_i\to\infty$; moreover,

$$|v^{n_i+1}-v^{n_i}|^2\to 0, \qquad |p^{n_i+1}-p^{n_i}|^2\to 0.$$

Consider inequalities (4.6) and (4.7) for all $n_i\to\infty$; passing to the limit, we obtain

$$\langle\nabla\Phi_w(v',v')+\nabla g^\top(v')p', w-v'\rangle\geq 0, \qquad p'=\pi_+(p'+\alpha g(v')).$$

Since these relations are equivalent to (4.2), we have $v'=v^*\in\Omega^*$ and $p'=p^*\geq 0$; i.e., any limit point of the sequence v^n, p^n is a solution to the problem. Since the quantity $|v^n-v^*|+|p^n-p^*|$ is monotonically decreasing, the limit point is unique; i.e., we have the convergence $v^n\to v^*$, $p^n\to p^*$ as $n\to\infty$. This completes the proof. □

This proof can be easily generalized if one has to deal with approximate solutions to a regularized problem, and the functions $\Phi(v,w)$ and $g(w)$ are given approximately.

5. THE PREDICTION METHOD OF PROXIMAL REGULARIZATION

Method (4.4), (4.5) is based on the modified Lagrange function. This ensures the convergence of the method. In many cases, however, this circumstance results in the loss of the decomposition properties; i.e., if the original problem has a block-separable structure (the normalized form of an n-person game always has such a structure), which allows one to decompose the original problem into independent subproblems, then the use of the modified Lagrange function results in the loss of this structure. On the other hand, if one uses the conventional Lagrange function, rather than the modified one, then the block-separable structure of the problem is preserved, because the conventional Lagrange function is a mere linear convolution of the objective function and the functional constraints. The last assertion means that if, in the iteration methods, one uses the conventional Lagrange function (rather than the modified one), then, at every iteration, the auxiliary optimization problem is decomposed into several independent subproblems of smaller dimension. This circumstance is of importance for game problems, because, as a rule, they have large dimensions.

In this section, we consider an analogue of method (4.4), (4.5) based on the conventional Lagrange function. Assume that we know the approximation v^n, p^n; then, the next approximation v^{n+1}, p^{n+1} can be found by the formulas

$$
\begin{aligned}
&\bar{p}^n = \pi_+(p^n + \alpha g(v^n)),\\
&v^{n+1} = \text{argmin}\left\{\frac{1}{2}|w - v^n|^2 + \alpha L(v^{n+1}, w, \bar{p}^n) \mid w \in \Omega\right\},\\
&p^{n+1} = \pi_+(p^n + \alpha g(v^{n+1})),
\end{aligned} \tag{5.1}
$$

where

$$L(v, w, p) = \Phi(v, w) + \langle p, g(w)\rangle.$$

Further, we use the following inequality:

$$\frac{1}{2}|z^* - x|^2 + \alpha f(z^*) \le \frac{1}{2}|z - x|^2 + \alpha f(z) - \frac{1}{2}|z - x^*|^2, \tag{5.2}$$

where $z \in Q$ and z^* is the minimum point of the function $|z - x|^2/2 + \alpha f(z)$ on the set Q for a fixed vector x. This inequality is valid for any convex functions that are not necessarily differentiable (see [2]).

Let us represent (5.1) as variational inequalities. We write the second of equations (5.1) as (5.2):

$$
\begin{aligned}
&\frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(v^{n+1}, v^{n+1}) + \alpha\langle\bar{p}^n, g(v^{n+1})\rangle \le\\
\le\ &\frac{1}{2}|w - v^n|^2 + \alpha\Phi(v^{n+1}, w) + \alpha\langle\bar{p}^n, g(w)\rangle - \frac{1}{2}|v^{n+1} - w|^2 \quad \forall w \in \Omega,
\end{aligned} \tag{5.3}
$$

the first and third equations (5.1) can be written as

$$\langle\bar{p}^n - p^n - \alpha g(v^n), p - \bar{p}^n\rangle \ge 0 \qquad \forall p \ge 0, \tag{5.4}$$

$$\langle p^{n+1} - p^n - \alpha g(v^{n+1}), p - p^{n+1}\rangle \ge 0 \qquad \forall p \ge 0. \tag{5.5}$$

Assume that the vector-valued function $g(w)$ satisfies the Lipschitz condition

$$|g(w + h) - g(w)| \le |g|\,|h| \tag{5.6}$$

for all $w, w + h \in \Omega$, where $|g|$ is a constant. Let us evaluate the difference between the two vectors $\bar{p}^n$ and p^{n+1}. Taking into account (5.6), we derive from (5.1) the following estimate:

$$|\bar{p}^n - p^{n+1}| \le \alpha|g(v^n) - g(v^{n+1})| \le \alpha|g|\,|v^n - v^{n+1}|. \tag{5.7}$$

Let us prove that (5.1) converges to an equilibrium solution the problem monotonically in the norm.

Theorem 2. *If the set of solutions to* (1.1) *is nonempty ami satisfies condition* (2.5) *for all* $w \in \Omega$, *the objective function* $\Phi(v, w)$ *is continuous with respect to* v *and convex with respect to* w *for every* $v \in \Omega$, *the convex vector-valued function* $g(w)$ *satisfies condition* (5.6), *and* $\Omega \subseteq \mathbb{R}^n$ *is a convex closed set, then the sequence* v^n *generated by* (5.1) *with the parameter* $0 < \alpha < (\sqrt{2}|g|)^{-1}$ *converges to an equilibrium solution monotonically in the norm, i.e.,* $v^n \to v^* \in \Omega^*$ *as* $n \to \infty$.

Proof. Setting $w = v^*$ in (5.3), we obtain

$$\begin{aligned}
&\frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(v^{n+1}, v^{n+1}) + \alpha\langle \bar{p}^n, g(v^{n+1})\rangle \leq \\
\leq \; &\frac{1}{2}|v^* - v^n|^2 + \alpha\Phi(v^{n+1}, v^*) + \alpha\langle \bar{p}^n, g(v^*)\rangle - \frac{1}{2}|v^{n+1} - v^*|^2
\end{aligned}$$

and setting $w = v^{n+1}$ in (4.2), we obtain

$$\Phi(v^*, v^*) + \langle p^*, g(v^*)\rangle \leq \Phi(v^*, v^{n+1}) + \langle p^*, g(v^{n+1})\rangle.$$

Now, we sum these two inequalities:

$$\begin{aligned}
&\frac{1}{2}|v^{n+1} - v^*|^2 + \frac{1}{2}|v^{n+1} - v^n|^2 + \\
+ \; &\alpha[\Phi(v^{n+1}, v^{n+1}) - \Phi(v^{n+1}, v^*) - \Phi(v^*, v^{n+1}) + \Phi(v^*, v^*)] + \\
+ \; &\alpha\langle \bar{p}^n - p^*, g(v^{n+1}) - g(v^*)\rangle \leq \frac{1}{2}|v^* - v^n|^2.
\end{aligned} \tag{5.8}$$

Consider inequalities (5.4) and (5.5). Let us set $p = p^*$ in (5.5)

$$\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle - \alpha\langle g(v^{n+1}), p^* - p^{n+1}\rangle \geq 0 \tag{5.9}$$

and $p = p^{n+1}$ in (5.4)

$$\begin{aligned}
\langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle \; &+ \; \alpha\langle g(v^{n+1}) - g(v^n), p^{n+1} - \bar{p}^n\rangle - \\
&- \; \alpha\langle g(^{n+1}), p^{n+1} - \bar{p}^n\rangle \geq 0,
\end{aligned} \tag{5.10}$$

Evaluating the second summand in this inequality with the help of (5.6) and (5.7) and adding inequalities (5.9) and (5.10), we obtain the following inequality:

$$\begin{aligned}
&\langle p^{n+1} - p^n, p^* - p^{n+1}\rangle + \langle \bar{p}^n - p^n, p^{n+1} - \bar{p}^n\rangle + \\
+ \; &\alpha^2|g|^2\,|v^{n+1} - v^n|^2 - \alpha\langle g(v^{n+1}), p^* - \bar{p}^n\rangle \geq 0.
\end{aligned}$$

Using identity (4.13), we expand the first two scalar products in a sum of squares:

$$\begin{aligned}
&\frac{1}{2}|p^{n+1} - p^*|^2 + \frac{1}{2}|p^{n+1} - \bar{p}^n|^2 + \frac{1}{2}|\bar{p}^n - p^n|^2 - \\
- \; &\alpha^2|g|^2\,|v^{n+1} - v^n|^2 + \alpha\langle g(v^{n+1}), p^* - \bar{p}^n\rangle \leq \frac{1}{2}|p^n - p^*|^2.
\end{aligned} \tag{5.11}$$

Then, we sum (5.8) and (5.11). Using (2.5) and the relations $\langle \bar{p}^n, g(v^*)\rangle \leq 0$and $\langle p^*, g(v^*)\rangle = 0$, we obtain

$$\begin{aligned}
&|v^{n+1} - v^*|^2 + (1 - 2\alpha^2|g|^2)|v^{n+1} - v^*|^2 + |p^{n+1} - p^*|^2 + \\
+ \; &|p^{n+1} - \bar{p}^n|^2 + |\bar{p}^n - p^n|^2 \leq |v^n - v^*|^2 + |p^n - p^*|^2.
\end{aligned} \tag{5.12}$$

Inequality (5.12) is a direct analogue of (4.14); hence, for $0 < \alpha < (\sqrt{2}|g|)^{-1}$, the proof of the theorem can be completed by using the proof scheme of Theorem 2. □

This proof can be generalized for the case when the auxiliary regularized solution is calculated approximately, and the initial data of the problem are also given approximately.

6. PREDICTION PROXIMAL-TYPE METHOD

In the preceding sections, we considered implicit iteration schemes, i.e., the schemes that the variables for which the auxiliary equations are solved at every iteration step both on the right- and left-hand sides of these equations. Hence, at every iteration, one has to solve a regularized equilibrium auxiliary problem, which is by no means easy. Therefore, the question arises whether it is possible to organize the calculation so that, at every iteration, the auxiliary problem would consist of one or several conventional problems of the minimization of a strongly convex function on a simple set. The answer to this question is positive. To verify that, consider one of several possible proximal iteration schemes. Indeed, let v^0, p^0 be an initial approximation; then, the next approximations can be calculated by the following recurrence formulas:

$$\begin{aligned}
\bar{p}^n &= \pi_+(p^n + \alpha g(v^n)), \\
\bar{u}^n &= \operatorname{argmin}\left\{\frac{1}{2}|w - v^n|^2 + \alpha L(v^n, w, \bar{p}^n) \mid w \in \Omega\right\}, \\
v^{n+1} &= \operatorname{argmin}\left\{\frac{1}{2}1|w - v^n|^2 + \alpha L(\bar{u}^n, w, \bar{p}^n) \mid w \in \Omega\right\}, \\
p^{n+1} &= \pi_+(p^n + \alpha g(\bar{u}^n)),
\end{aligned} \tag{6.1}$$

where

$$L(v, w, p) = \Phi(v, w) + \langle p, g(w)\rangle.$$

Let us represent this process as a set of variational inequalities. According to the definition of the projection operator, we can write the first and the fourth equations of (6.1) as

$$\langle \bar{p}^n - p^n - \alpha g(v^n), p - \bar{p}^n\rangle \geq 0 \quad \forall p \geq 0 \tag{6.2}$$

and

$$\langle p^{n+1} - p^n - \alpha g(\bar{u}^n), p - p^{n+1}\rangle \geq 0 \quad \forall p \geq 0. \tag{6.3}$$

respectively.

Writing the second and the third equations as (5.2), we obtain

$$\begin{aligned}
& \frac{1}{2}|\bar{u}^n - v^n|^2 + \alpha\Phi(v^n, \bar{u}^n) + \alpha\langle \bar{p}^n, g(\bar{u}^n)\rangle \leq \\
\leq\ & \frac{1}{2}|w - v^n|^2 + \alpha\Phi(v^n, w) + \alpha\langle \bar{p}^n, g(w)\rangle - \frac{1}{2}|w - \bar{u}^n|^2 \qquad \forall w \in \Omega
\end{aligned} \tag{6.4}$$

and

$$\begin{aligned}
& \frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(\bar{u}^n, v^{n+1}) + \alpha\langle \bar{p}^n, g(v^{n+1})\rangle \leq \\
\leq\ & \frac{1}{2}|w - v^n|^2 + \alpha\Phi(\bar{u}^n, w) + \alpha\langle \bar{p}^n, g(w)\rangle - \frac{1}{2}|v^{n+1} - w|^2 \quad \forall w \in \Omega.
\end{aligned} \tag{6.5}$$

Throughout the rest of this paper, we assume that the function $\Phi(v, w)$ satisfies the following Lipschitz-type condition:

$$|[\Phi(w + h, v + k) - \Phi(w + h, v)] - [\Phi(w, v + k) - \Phi(w, v)]| \leq |\Phi|\ |h|\ |k| \tag{6.6}$$

for all $w, w + h \in \Omega$ and $v, v + k \in \Omega$, where $|\Phi|$ is a constant. A class of functions that satisfy this condition is nonempty [2]. Moreover, the vector-valued function $g(w)$ satisfies the condition

$$|g(w + h) - g(w)| \leq |g|\,|h| \tag{6.7}$$

for all $w, w+h \in \Omega$, where $|g|$ is a constant.

Let us evaluate the difference between the vectors v^{n+1} and $\bar{u}^n$. For this purpose, we put $w = v^{n+1}$ in (6.4) and $w = \bar{u}^n$ in (6.5); this yields

$$\begin{aligned} &\frac{1}{2}|\bar{u}^n - v^n|^2 + \alpha\Phi(v^n, \bar{u}^n) + \alpha\langle \bar{p}^n, g(\bar{u}^n)\rangle \le \\ \le\ &\frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(v^n, v^{n+1}) + \alpha\langle \bar{p}^n, g(v^{n+1})\rangle - \frac{1}{2}|v^{n+1} - \bar{u}^n|^2, \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(\bar{u}^n, v^{n+1}) + \alpha\langle \bar{p}^n, g(v^{n+1})\rangle \le \\ \le\ &\frac{1}{2}|\bar{u}^n - v^n|^2 + \alpha\Phi(\bar{u}^n, \bar{u}^n) + \alpha\langle \bar{p}^n, g(\bar{u}^n)\rangle - \frac{1}{2}|\bar{u}^n - v^{n+1}|^2. \end{aligned}$$

Let us sum up the two inequalities thus obtained:

$$|\bar{u}^n - v^{n+1}|^2 + \alpha[\Phi(v^n, \bar{u}^n) - \Phi(v^n, v^{n+1}) - \Phi(\bar{u}^n, \bar{u}^n) + \Phi(\bar{u}^n, v^{n+1})] \le 0.$$

Taking into account (6.6), we obtain

$$|\bar{u}^n - v^{n+1}| \le \alpha|\Phi|\,|v^n - \bar{u}^n|. \tag{6.8}$$

Now, let us show that (6.1) converges to an equilibrium solution monotonically in the norm.

Theorem 3. *If the set of solutions to* (1.1) *is nonempty and satisfies condition* (2.5) *for all $w \in \Omega$, the objective function $\Phi(v, w)$ is continuous with respect to v and convex with respect to w for every $v \in \Omega$, $\Omega \subseteq \mathbb{R}^n$ is a convex closed set, and the functions $\Phi(v, w)$ and $g(w)$ are convex with respect to w and satisfy conditions* (6.6) *and* (6.7), *then the sequence v^n generated by* (6.1) *with the parameter $0 < \alpha < (\sqrt{2(|\Phi|^2 + |g|^2)})^{-1}$ converges to an equilibrium solution monotonically in the norm, i.e., $v^n \to v^* \in \Omega^*$ as $n \to \infty$.*

Proof. Setting $w = v^*$ in (6.5), we obtain

$$\begin{aligned} &\frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(\bar{u}^n, v^{n+1}) + \alpha\langle \bar{p}^n, g(v^{n+1})\rangle \le \\ \le\ &\frac{1}{2}|v^* - v^n|^2 + \alpha\Phi(\bar{u}^n, v^*) + \alpha\langle \bar{p}^n, g(v^*)\rangle - \frac{1}{2}|v^{n+1} - v^*|^2. \end{aligned}$$

Hence,

$$\begin{aligned} &\frac{1}{2}|v^{n+1} - v^*|^2 + \frac{1}{2}|v^{n+1} - v^n|^2 + \alpha[\Phi(\bar{u}^n, v^{n+1}) - \Phi(\bar{u}^n, v^*)] + \\ +\ &\alpha\langle \bar{p}^n, g(v^{n+1}) - g(v^*)\rangle \le \frac{1}{2}|v^n - v^*|^2. \end{aligned} \tag{6.9}$$

Setting $w = v^{n+1}$ in (6.4), we infer that

$$\begin{aligned} &\frac{1}{2}|\bar{u}^n - v^n|^2 + \alpha\Phi(v^n, \bar{u}^n) + \alpha\langle \bar{p}^n, g(\bar{u}^n)\rangle \le \\ \le\ &\frac{1}{2}|v^{n+1} - v^n|^2 + \alpha\Phi(v^n, v^{n+1}) + \alpha\langle \bar{p}^n, g(v^{n+1})\rangle - \frac{1}{2}|v^{n+1} - \bar{u}^n|^2. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{1}{2}|v^{n+1}-\bar{u}^n|^2 &+ \frac{1}{2}|\bar{u}^n-v^n|^2+\alpha[\Phi(v^n,\bar{u}^n)-\Phi(v^n,v^{n+1})]+ \\ &+ \alpha\langle\bar{p}^n, g(\bar{u}^n)-g(v^{n+1})\rangle \le \frac{1}{2}|v^{n+1}-v^n|^2. \end{aligned} \tag{6.10}$$

Summing inequalities (6.9) and (6.10), we obtain

$$\begin{aligned} \frac{1}{2}|v^{n+1}-v^*|^2 &+ \frac{1}{2}|v^{n+1}-\bar{u}^n|^2+\frac{1}{2}|\bar{u}^n-v^n|^2+\alpha[\Phi(\bar{u}^n,\bar{u}^n)-\Phi(\bar{u}^n,v^*)]- \\ &- \alpha[\Phi(\bar{u}^n,\bar{u}^n)-\Phi(\bar{u}^n,v^{n+1})-\Phi(v^n,\bar{u}^n)+\Phi(v^n,v^{n+1})]+ \\ &+ \alpha\langle\bar{p}^n, g(\bar{u}^n)-g(v^*)\rangle \le \frac{1}{2}|v^n-v^*|^2. \end{aligned} \tag{6.11}$$

Setting $w=\bar{u}^n$ in (4.2), we write the inequality thus obtained as

$$\Phi(v^*,v^*)+\langle p^*, g(v^*)\rangle \le \Phi(v^*,\bar{u}^n)+\langle p^*, g(\bar{u}^n)\rangle.$$

We add this inequality to (6.11). Taking into account (6.6) – (6.8), we obtain

$$\begin{aligned} \frac{1}{2}|v^{n+1}-v^*|^2 &+ \frac{1}{2}|v^{n+1}-\bar{u}^n|^2+\left(\frac{1}{2}-\alpha^2|\Phi|^2\right)|\bar{u}^n-v^n|^2+ \\ &+ \alpha[\Phi(\bar{u}^n,\bar{u}^n)-\Phi(\bar{u}^n,v^*)-\Phi(v^*,\bar{u}^n)+\Phi(v^*,v^*)]+ \\ &+ \alpha\langle\bar{p}^n-p^*, g(\bar{u}^n)-g(v^*)\rangle \le |v^n-v^*|^2. \end{aligned} \tag{6.12}$$

Consider (6.2) and (6.3). Setting $p=p^*$ in (6.3), we obtain

$$\langle p^{n+1}-p^n, p^*-p^{n+1}\rangle-\alpha\langle g(\bar{u}^n), p^*-p^{n+1}\rangle \ge 0, \tag{6.13}$$

and setting $p=p^{n+1}$ in (6.2), we obtain

$$\begin{aligned} \langle\bar{p}^n-p^n, p^{n+1}-\bar{p}^n\rangle &+ \alpha\langle g(\bar{u}^n)-g(v^n), p^{n+1}-\bar{p}^n\rangle- \\ &- \alpha\langle g(\bar{u}^n), p^{n+1}-\bar{p}^n\rangle \ge 0. \end{aligned} \tag{6.14}$$

We evaluate the second summand in this inequality using (6.7) and then sum up inequalities (6.13) and (6.14):

$$\begin{aligned} &\langle p^{n+1}-p^n, p^*-p^{n+1}\rangle+\langle\bar{p}^n-p^n, p^{n+1}-\bar{p}^n\rangle+ \\ +\; &\alpha^2|g|^2\,|\bar{u}^n-v^n|^2-\alpha\langle g(\bar{u}^n), p^*-\bar{p}^n\rangle \ge 0, \end{aligned}$$

Using (4.13), we expand the first two scalar products in a sum of squares:

$$\begin{aligned} \frac{1}{2}|p^{n+1}-p^*|^2 &+ \frac{1}{2}|p^{n+1}-\bar{p}^n|^2+\frac{1}{2}|\bar{p}^n-p^n|^2-\alpha^2|g|^2\,|\bar{u}^n-v^n|^2+ \\ &+ \alpha\langle g(\bar{u}^n), p^*-\bar{p}^n\rangle \le \frac{1}{2}|p^n-p^*|^2. \end{aligned} \tag{6.15}$$

Then, we sum up (6.12) and (6.15). Taking into account the formulas $\langle p^{n+1}, g(v^*)\rangle \le 0$, $\langle p^*, g(v^*)\rangle = 0$, and (2.5), we obtain

$$\begin{aligned} &\frac{1}{2}|v^{n+1}-v^*|^2+\frac{1}{2}|p^{n+1}-p^*|^2+\frac{1}{2}|v^{n+1}-\bar{u}^n|^2+ \\ +\; &\left[\left(\frac{1}{2}-\alpha^2(|\Phi|^2+|g|^2)\right)\right]|\bar{u}^n-v^n|^2+\frac{1}{2}|p^{n+1}-\bar{p}^n|^2+\frac{1}{2}|\bar{p}^n-p^n|^2 \le \\ \le\; &\frac{1}{2}|v^n-v^*|^2+\frac{1}{2}|p^n-p^*|^2. \end{aligned} \tag{6.16}$$

Summing (6.16) from $n = 0$ to $n = N$, we obtain

$$\begin{aligned} |v^{N+1} - v^*|^2 &+ |p^{N+1} - p^*|^2 + 2d \sum_{k=0}^{k=N} |\bar{u}^k - v^k|^2 + \sum_{k=0}^{k=N} |v^{k+1} - \bar{u}^k|^2 + \\ &+ \sum_{k=0}^{k=N} |p^{k+1} - \bar{p}^k|^2 + \sum_{k=0}^{k=N} |\bar{p}^k - p^k|^2 \leq |v^0 - v^*|^2 + |p^0 - p^*|^2, \end{aligned}$$

where $d = 1/2 - \alpha^2(|\Phi|^2 + |g|^2) > 0$. The inequality thus obtained implies the boundedness of the trajectory

$$|v^{N+1} - v^*|^2 + |p^{N+1} - p^*|^2 \leq |v^0 - v^*|^2 + |p^0 - p^*|^2,$$

and the convergence of the series

$$\sum_{k=0}^{\infty} |\bar{u}^k - v^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |v^{k+1} - \bar{u}^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |p^{k+1} - \bar{p}^k|^2 < \infty, \quad \sum_{k=0}^{\infty} |\bar{p}^k - p^k|^2 < \infty,$$

hence,

$$|\bar{u}^n - v^n|^2 \to 0, \quad |v^{n+1} - \bar{u}^n|^2 \to 0, \quad |p^{n+1} - \bar{p}^n|^2 \to 0, \quad |\bar{p}^n - p^n|^2 \to 0, \qquad n \to \infty.$$

Since the sequence v^n, p^n is bounded, there exists an element v', p' such that $v^{n_i} \to v'$ and $p^{n_i} \to p'$ as $n_i \to \infty$; moreover,

$$|v^{n_i+1} - \bar{u}^{n_i}|^2 \to 0, \quad |\bar{u}^{n_i} - v^{n_i}|^2 \to 0, \quad |p^{n_i+1} - \bar{p}^{n_i}|^2 \to 0, \quad |\bar{p}^{n_i} - p^{n_i}|^2 \to 0.$$

Consider inequalities (6.3) and (6.5) for all $n_i \to \infty$; passing to the limit yields

$$p' = \pi_+(p' + \alpha g(v')), \qquad \Phi(v', v') + \langle p', g(v') \rangle \leq \Phi(v', w) + \langle p', g(w) \rangle$$

Since these relations are equivalent to (4.3), we have $v' = v^* \in \Omega^*$ and $p' = p^* \geq 0$; i.e., any limit point of the sequence v^n, p^n is a solution to the problem. Since $|v^n - v^*| + |p^n - p^*|$ is monotonically decreasing, there exists precisely one unique limit point, i.e., $v^n \to v^*$ and $p^n \to p^*$ as $n \to \infty$. This completes the proof. □

Condition (2.5) plays an important role in the proof of the convergence of the methods (4.4), (5.1), and (6.1). This condition is more restrictive than (2.3). For some problems (e.g., the Cournot diopoly), condition (2.3) may be fulfilled, but condition (2.5) may not. Nevertheless, the above methods can be applied in this case if one uses the following consideration. The left-hand sides of inequalities (4.9), (5.8), and (6.11) contain two terms for which some problems arise. The first term, namely, the summand $\Phi(\bar{u}^n, v^*) - \Phi(\bar{u}^n, \bar{u}^n)$, hardly causes any trouble, because it can be estimated with the help of (2.3). The situation with the second term, namely, $\langle p^{n+1}, g(v^*) - g(\bar{u}^n) \rangle$, is somewhat more complicated, because this term must be nonpositive. Therefore, we apply the estimate $\langle p^{n+1}, g(v^*) - g(\bar{u}^n) \rangle \leq 0$, using the inequality $\langle p^{n+1}, g(v^*) \rangle \leq 0$ and assuming that the quantity $g(\bar{u}^n)$ (or, better still, $\langle p^{n+1}, g(\bar{u}^n) \rangle$) is nonnegative, because any of the methods under consideration is external with respect to the admissible domain $D = \{w \mid g(w) \leq 0\}$. This means that, if the initial approximation v^0 satisfies the condition $g(v^0) \geq 0$, then all subsequent approximations satisfy it with high probability, i.e., $g(v^n) \geq 0$. Moreover, it is possible to verify the last condition when implementing the method.

Taking into account the above consideration, we can write the analogues of inequality (4.15) in Theorems 2 and 3 as follows:

$$|v^{n+1} - v^*|^2 + d_1|\bar{u}^n - v^n|^2 + d_2|v^{n+1} - \bar{u}^n|^2 \le |v^n - v^*|^2.$$

This inequality implies that the sequence v^n converges to an equilibrium solution $v^* \in \Omega^*$. To prove this statement, it is sufficient to use the above proofs, beginning with formulas (4.9), (5.8), and (6.11).

ACKNOWLEDGMENTS

This work was supported by the Russian Foundation for Basic Research, project no. 96-01-01046.

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