CANTOR'S DIAGONAL ARGUMENT: A NEW ASPECT.

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Abstract. – In the paper, Cantor's diagonal proof of the theorem about the cardinality of power set, $|X| < |P(X)|$, is analyzed. It is shown first that a key point of the proof is an *explicit* usage of the counter-example method. It means that an *only* counter-example (Cantor's new element of P(X) not belonging to a mapping of X onto P(X)) is *sufficient* in order to formally disprove a *common* statement (the assumption of Cantor's proof that there is a mapping of X onto P(X) including *all* elements from P(X)), but a total number of all possible counter-examples (a cardinality of P(X)) plays no role in such a disproof. In addition Cantor's conclusion in the form $|X| < |P(X)|$ is *deduced* from the *fact* that the difference between *infinite* sets, P(X) and X, amounts to *one* element, that is such conclusion contradicts fatally the main property of *infinite* sets. So, it takes place the following unique situation: the formal logic of Cantor's proof is unobjectionable, but the proof itself has no relation to and does not use quantitative properties, i.e., a number of elements or a cardinality, of the set, $|P(X)|$. It is proved as well that if to suppose that a set of all possible Cantor's counter-examples is infinite, then the Cantor argument leads to an infinite "implication" which does not allow to disprove the assumption, $|X| = |P(X)|$, i.e., makes Cantor's statement, $|X| < |P(X)|$, *unprovable* within the framework of just *traditional* Cantor's proof.

The outstanding meta-mathematician, W.Hodges, in his famous paper "*An Editor Recalls Some Hopeless Papers*" [**5**], gives a brilliant analysis of main "attacks against Cantor's diagonal argument". In particular, he emphasizes that "all the Cantor critics attack" the "elementary" version of the argument which uses a matrix representation of a sequence of decimal real numbers. In this connection, W.Hodges expresses a perplexity apropos of the fact that "none of the authors showed any knowledge of Cantor's theorem about the cardinalities of power sets."

Just this remark of the outstanding meta-mathematician prompted the idea to analyze in more details the common power-set version of Cantor's Theorem, which is really much more "short and lucid" than the traditional matrix version.

So, consider the Cantor theorem on the cardinality of a power-set [**2,3**] and its *traditional* diagonal proof in its modern set-theoretical ZF-form [**4**].

Here P(X) is a power-set and $|X|$ is a cardinality of an arbitrary set X, and, for short, RAA = Reductio and Absurdum, CDM = Cantor's Diagonal Method, AD-element = Anti-Diagonal element produced by CDM.

CANTOR'S THEOREM (1890). $|X| < |P(X)|$.

PROOF (by RAA-method). It's obvious that $|X| \leq |P(X)|$.

Assume that φ maps X onto P(X). Define a new subset of X as follows: $X^*=\{x \in X \mid x \notin \varphi(x)\}$. Then $X^* \subseteq X$, and if $X^* = \varphi(y)$ for some $y \in X$, then $y \in X^* \rightarrow y \notin \varphi(y) \rightarrow y \notin X^*$ and $y \notin X^* \rightarrow y \in \varphi(y) \rightarrow y \in X^*$. The last is impossible. Q.E.D.

To begin an analysis of the proof, remind of some fundamental statements of modern set theory.

DEFINITION-1 (Cantor). A set Z is *infinite* iff it's *equivalent* to its *own* subset [**2,3**], [**6**] (p. 20).

Directly from this definition, the following statement follows [**1**] (pp. 27-28), [**6**] (p. 20).

LEMMA-1. If a difference between numbers of elements of two *infinite* sets, say Z_1 and Z_2, is *finite*, then the sets, Z_1 and Z_2, are *equivalent*, i.e., $|Z_1| = |Z_2|$.

Now we shall prove two theorems elucidating some "hidden" logical peculiarities of the "short and lucid" Cantor's diagonal argument.

THEOREM 1. Cantor's conclusion $|X| < |P(X)|$ contradicts Lemma 1 [**8,9,12,13**].

PROOF. Following Cantor, assume that φ maps X onto P(X), i.e., $|X| = |P(X)|$.

Represent the set P(X) as the sum, $P(X) = P_1+P_2$, where P_1 is a set of all elements from P(X), which really belong to φ, and P_2 is a complement to P_1 in P(X): $P_2 = P(X) – P_1$. It's obvious that, by virtue of RAA-assumption, $|P_1|=|X|$, and P_2 is an *empty* set.

For the given φ Cantor defines $X^*=\{x \in X \mid x \notin \varphi(x)\}$ which is an element of P(X), but does not belong to φ, and proves that P_2 is not *empty*.

A key point and a specific peculiarity of Cantor's RAA-proof is an *explicit* usage of the *counter-example method*. Indeed, from the point of view of classical logic and classical mathematics, Cantor's anti-diagonal AD-element X* is a counter-example, disproving the *common* statement **B** = «a *given* mapping φ includes *all* elements from P(X)».

From the proven falsity of the statement **B** (just by means of the counter-example method) it follows (by classical *modus tollens* rule) that RAA-assumption, $|X| = |P(X)|$, is *false*, and the proven inequality $|X| \neq |P(X)|$, together with the obvious inequality, $|X| \leq |P(X)|$, leads to the finale Cantor statement $|X| < |P(X)|$.

So, from the point of view of formal logic, Cantor's RAA-proof is blameless and irrefutable. However, the peculiarity just of the counter-example method consists in that an *only* counter-example is *sufficient* in order *to disprove* a *common* statement, but a *total* (finite or even infinite) number of all possible counter-examples does not play here any role (see, e.g., the famous classical example of the application of counter-example method in mathematics in [**7**]).

However, it's obvious that for a *given* (*fixed*) mapping φ, the Cantor's 'diagonal' method is able to produce the *only*, *unique* element X* of the set P(X), *not belonging* to the *given* φ, i.e., $|P_2| = 1$. It means that Cantor's conclusion $|X|<|P(X)|$ is based on the *fact* that *infinite* set P(X) has *only one element* greater than the *infinite* set X, i.e., $|P(X)| - |X| =1$. It's obvious that such "a set theoretical ground" for Cantor's conclusion $|X|<|P(X)|$ contradicts fatally to the set-theoretical Lemma 1, according to which, the equality $|P(X)| - |X| =1$ entails $|X| = |P(X)|$. Q.E.D.

THEOREM 2. The Cantor *inequality* $|X|<|P(X)|$ is *unprovable* [**10 - 13**]

PROOF. The *only* reason to state that $|X|<|P(X)|$ is Cantor's Theorem above. Therefore in order *to prove* our Theorem 2 it's *sufficient to prove* that traditional Cantor's diagonal proof *does not prove* the statement $|X|<|P(X)|$.

Toward this end, consider again the *traditional* Cantor's proof.

CANTOR'S THEOREM. $|X| < |P(X)|$.

PROOF-1. Assume that φ maps X onto P(X), i.e., $|X| = |P(X)|$.

Represent the set P(X) as the sum, $P(X) = P_{11}+P_{12}$, where P_{11} is a set of all elements from P(X), which really belong to φ, and P_{12} is a complement to P_{11} in P(X): $P_{12} = P(X) – P_{11}$. It's obvious that, by virtue of the RAA-assumption, $|P_{11}|=|X|$, and P_{12} is an *empty* set.

For the given φ Cantor defines $X^*=\{x \in X \mid x \notin \varphi(x)\}$ which does not belong to φ, and proves that P_{12} is not *empty*. But now we shall admit that changing the initial mapping φ, Cantor's diagonal definition is able to produce an *infinite* set P_{12} of *new* AD-elements from P(X), not belonging to the *initial* mapping φ, i.e., not belonging to P_{11}.

The following two cases are possible.

(i) $|P_{12}| = |X|$. If that is so, then $|P(X)| = |P_{11} + P_{12}| =|X|$, and therefore to disprove the RAA-assumption $|X| = |P(X)|$ is impossible from the point of view just of axiomatic set theory.

(ii) $|P_{12}| > |X|$. However, since Cantor's proof so far is not completed the very existence of a cardinality which is greater than $|X|$ is so far not proven, and therefore a *hypothetical* statement,

$|P_{12}| > |X|$, *must be proved*. It can be done by the only way – by means of the CDM, i.e., one must now prove the *initial* Cantor's Theorem with the new *symbol* P_{12} instead of the old *symbol* P(X).

CANTOR'S THEOREM. $|X| < |P_{12}|$.

PROOF-2. Assume that φ_1 maps X onto $|P_{12}|$, i.e., $|X| = |P_{12}|$. The application of the CDM to φ_1, produces an *infinite* set of elements from P_{12} which don't belong to the φ_1.

Represent the set P_{12} as a sum $P_{12} = P_{21} + P_{22}$, where P_{21} is a set of elements of P_{12} really included in the φ_1, but the set P_{22} is a complement to P_{21} in P_{12}: $P_{22} = P_{12} – P_{21}$. It's obvious that $|P_{21}|=|X|$, and the complement P_{22} contains *all* AD-elements which can be produced by the CDM-application to the φ_1. The last means that now the veritable cardinality of P_{12} is defined by and is equal to the cardinality of the complement P_{22}, i.e., $|P_{12}| = |P_{22}|$.

Consider the following two cases.

(i) $|P_{22}| = |X|$. If that is so, then $|P_{12}| = |P_{21} + P_{22}| =|X|$, and therefore it's impossible to disprove the RAA-assumption, $|X| = |P_{12}|$, from the point of view just of axiomatic set theory.

(ii) $|P_{22}| > |X|$. However, since Cantor's proof so far is not completed the very existence of a cardinality which is greater than |X| is so far not proven, and therefore a *hypothetical* statement, $|P_{22}| > |X|$, *must be proved*. It can be done by the only way – by means of the CDM, i.e., one must now prove the *initial* Cantor's Theorem with the new *symbol* P_{22} instead of the old *symbol* P_{12}.

And so on ad infinitum.

Thus, the traditional Cantor "proof" of the Theorem about the cardinality of a power set, from the *set-theoretical* point of view (!), either isn't able to disprove the RAA-assumption, |X| = |P(X)|, or is reduced to the *infinite* system of "nested" ("embedded") proofs of the initial Cantor Theorem by the sequential replacement of the initial *symbol* P(X) by *symbols* P_{12}, P_{22}, P_{32}, …, that is to the following *non-finite, tautological, and quite senseless* "reasoning" (here $\mathbf{D_i}$ = «*it needs to prove that* $|X| < |P_{i2}|$ »):

$$\mathbf{D_1} \circledR \ \mathbf{D_2} \circledR \ \mathbf{D_3} \circledR \ \ldots \qquad (*)$$

It's obvious, that until the *potentially* infinite (obviously, *countable*) "reasoning" (*) is finished, the RAA-assumption |X| = |P(X)| of Cantor's RAA-proof is *irrefutable* from the set-theoretical points of view, and, consequently, the Cantor statement |X| < |P(X)| is *unprovable*. Q.E.D.

CONCLUSIONS.

1. For the first time the fact is revealed and explicitly formulated that the crucial point of Cantor's RAA-proof of the power-set theorem, stating that |X| < |P(X)|, is an *explicit* usage of the counter-example method.

2. The *formal* logic of Cantor's RAA-proof seems to be blameless and irrefutable. Indeed, from the point of view of the logic, an *only* counter-example (Cantor's AD-element $X^* \notin$ (**1**)) is *sufficient* in order to disprove the RAA-assumption |X| = |P(X)| of Cantor's diagonal proof. On the other hand, Cantor's conclusion in the form |X| < |P(X)| is *deduced* from the *fact* that the difference between *infinite* sets, P(X) and X, amounts to *one* element, i.e., |P(X)| - |X| =1. It is obvious that this Cantor conclusion contradicts fatally the main property of infinite sets in the form of Lemma 1, according to which from the fact |P(X)| - |X| =1 it follows that |X| = |P(X)|, i.e., that the sets X and |P(X)| are equivalent.

It means that there takes place the following unique (in all history of mathematics) situation: on the one hand, Cantor's conclusion, |X| < |P(X)|, is unobjectionable from the point of view of *formal logic*, but, on the other hand, the diagonal proof of the conclusion is based on the fact which contradicts Lemma 1, i.e., the Cantor conclusion that |X| < |P(X)| is wrong from the point of view of modern axiomatic (and 'non-naive') set theory. The proof itself has no relation to and does not use *quantitative* properties, i.e., a number of elements or a cardinality, of the set, P(X).

3. It is proved that in a common case when a set of all possible Cantor's counter-examples is, supposedly, *infinite*, the Cantor statement, $|X| < |P(X)|$, becomes *unprovable* within the framework of *traditional* Cantor's proof.

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